

6.3 Riemann Sums and Integral Notation

Name: _____ Class: _____ Date: _____

Total: 11 marks

KEY WORDS definite integral 定积分 • riemann sum 黎曼和 • antiderivative 原函数

Objective

Build the skills to answer exam questions on **Riemann sums and definite-integral notation** — connecting a limit of sums to $\int_a^b f dx$.

You must be able to:

- write a Riemann sum with **summation notation** $\sum$
- recognise the **definite integral** 定积分 as the limit of Riemann sums as $n \rightarrow \infty$
- read the meaning of the parts of $\int_a^b f(x) dx$

1 Worked examples

Study these first. Each one shows the method for a question type used later — follow the steps and you can do the Practice and Exam-style questions yourself.

■ Summation notation for a sum

A right Riemann sum with n equal pieces on $[a, b]$ has width $\Delta x = \frac{b-a}{n}$ and sample points $x_k = a + k \Delta x$:

$$\sum_{k=1}^n f(x_k) \Delta x.$$

■ The definite integral as a limit

Letting the rectangles become infinitely thin gives the exact area:

$$\int_a^b f(x) dx = \lim_{n \rightarrow \infty} \sum_{k=1}^n f(x_k) \Delta x.$$

The integral **is** this limit — a sum of infinitely many infinitely thin pieces.

■ Reading the notation

In $\int_a^b f(x) dx$: a and b are the **limits of integration**, $f(x)$ is the **integrand**, and dx marks the variable and the "width" of each piece. The result is a **number** (a signed area), not a function.

■ From a limit back to an integral

A limit like $\lim_{n \rightarrow \infty} \sum_{k=1}^n \sqrt{1 + \frac{3k}{n}} \cdot \frac{3}{n}$ is a right sum with $\Delta x = \frac{3}{n}$ on $[1, 4]$, so it equals

$$\int_1^4 \sqrt{x} dx.$$

2 Practice

Now apply the methods above.

2.1 For $\int_2^{10} f dx$ estimated with $n = 4$ equal subintervals, state Δx . [1]

2.2 Write, in summation notation, a right Riemann sum for $\int_0^2 f dx$ with n equal subintervals. [2]

2.3 In $\int_1^5 (3x + 2) dx$, name the integrand and the limits of integration. [2]

3 Exam-style questions

3.1 The definite integral $\int_a^b f dx$ is defined as [1]

- **A** an antiderivative of f
- **B** the limit of Riemann sums as $n \rightarrow \infty$
- **C** the slope of f
- **D** the average of $f(a)$ and $f(b)$

3.2 Consider $\lim_{n \rightarrow \infty} \sum_{k=1}^n \left(2 + \frac{4k}{n}\right)^2 \cdot \frac{4}{n}$.

(a) State the width Δx and the interval $[a, b]$. [2]

(b) Write the limit as a definite integral. [2]

3.3 A definite integral $\int_0^6 v(t) dt$ evaluates to a negative number. Explain what a **negative** value tells you about the signed area, in one sentence. [1]

Solutions

2.1 $\Delta x = \frac{10 - 2}{4} = 2.$

2.2 $\Delta x = \frac{2}{n}, x_k = \frac{2k}{n}; \text{sum} = \sum_{k=1}^n f\left(\frac{2k}{n}\right) \cdot \frac{2}{n}.$

2.3 Integrand $3x + 2$; limits of integration 1 and 5.

3.1 B — the definite integral is the limit of Riemann sums.

3.2 (a) $\Delta x = \frac{4}{n}$, so $b - a = 4$; the sample $2 + \frac{4k}{n}$ starts at 2, so $[a, b] = [2, 6]$. (b) $\int_2^6 x^2 dx.$

3.3 More signed area lies **below** the axis than above it over $[0, 6]$.