

6.10 Long Division and Completing the Square

Name: _____ Class: _____ Date: _____

Total: 14 marks

KEY WORDS complete the square 配方 • completing the square 配方法 • polynomial long division 多项式长除法

Objective

Build the skills to answer exam questions on **integrating with long division and completing the square** — rewriting a rational integrand into an integrable form.

You must be able to:

- use **polynomial long division** 多项式长除法 when the numerator's degree is $\geq$ the denominator's
- **complete the square** 配方 in a denominator to reach an **arctangent** form
- recognise $\int \frac{du}{u^2 + a^2} = \frac{1}{a} \arctan \frac{u}{a} + C$

1 Worked examples

Study these first. Each one shows the method for a question type used later — follow the steps and you can do the Practice and Exam-style questions yourself.

■ Long division first

$\int \frac{x^2}{x+1} dx$ is top-heavy. Divide: $\frac{x^2}{x+1} = x - 1 + \frac{1}{x+1}$. Then

$$\int \left(x - 1 + \frac{1}{x+1} \right) dx = \frac{x^2}{2} - x + \ln|x+1| + C.$$

■ A simple 1/x-type

$\int \frac{1}{x-3} dx = \ln|x-3| + C$ (substitution $u = x - 3$). Watch for these after dividing.

■ Completing the square

$\int \frac{1}{x^2 + 4x + 5} dx$: complete the square, $x^2 + 4x + 5 = (x+2)^2 + 1$. With $u = x + 2$,

$$\int \frac{du}{u^2 + 1} = \arctan(u) + C = \arctan(x+2) + C.$$

■ The arctangent template

$$\int \frac{1}{u^2 + a^2} du = \frac{1}{a} \arctan \frac{u}{a} + C. \text{ For } \int \frac{1}{x^2 + 9} dx = \frac{1}{3} \arctan \frac{x}{3} + C.$$

2 Practice

Now apply the methods above.

2.1 Use long division to rewrite $\frac{x+3}{x+1}$ as $1 + \frac{?}{x+1}$. [2]

2.2 Find $\int \frac{1}{x^2 + 16} dx$. [2]

2.3 Complete the square: write $x^2 + 6x + 13$ in the form $(x + a)^2 + b$. [2]

3 Exam-style questions

3.1 $\int \frac{1}{x^2 + 25} dx =$ [1]

- **A** $\arctan(x) + C$
 - **B** $\frac{1}{5} \arctan \frac{x}{5} + C$
 - **C** $\ln(x^2 + 25) + C$
 - **D** $5 \arctan(5x) + C$
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3.2 Find $\int \frac{x+2}{x+1} dx$. [3]

3.3 Find $\int \frac{1}{x^2 + 2x + 10} dx$ by completing the square. [4]

Solutions

2.1 $\frac{x+3}{x+1} = 1 + \frac{2}{x+1}.$

2.2 $\frac{1}{4} \arctan \frac{x}{4} + C.$

2.3 $(x+3)^2 + 4.$

3.1 B — with $a = 5$, $\int \frac{dx}{x^2+25} = \frac{1}{5} \arctan \frac{x}{5} + C.$

3.2 $\frac{x+2}{x+1} = 1 + \frac{1}{x+1}; \int \left(1 + \frac{1}{x+1}\right) dx = x + \ln|x+1| + C.$

3.3 $x^2 + 2x + 10 = (x+1)^2 + 9$; with $u = x+1$, $\int \frac{du}{u^2+9} = \frac{1}{3} \arctan \frac{u}{3} = \frac{1}{3} \arctan \frac{x+1}{3} + C.$