AP CALCULUS AB • EXERCISE SHEET

# 5.10 Introduction to Optimization

Name: ____________ Class: ________ Date: __________ **Total: 12 marks**

**KEY WORDS** constraint 约束 • optimization 最优化 • objective function 目标函数

## Objective

Build the skills to answer exam questions on **optimization 最优化** — turning a word problem into a function and finding its greatest or least value.

**You must be able to:**

- write a quantity to be maximized or minimized as a function of one variable (the **objective function 目标函数**)
- use a **constraint 约束** to eliminate the extra variable
- state the **domain** that makes physical sense for the problem

## 1 Worked examples

Study these first. Each one shows the method for a question type used later — follow the steps and you can do the Practice and Exam-style questions yourself.

### Naming the objective and the constraint

Optimization always has two ingredients: the **objective** (what you want biggest or smallest) and a **constraint** (a fixed condition). A farmer has 40 m of fence for a rectangular pen against a wall (no fence on the wall side).

- Objective: **area** $A = xy$ (maximize).
- Constraint: fence used $= x + 2y = 40$.

### Reducing to one variable

Solve the constraint for one variable and substitute. From $x + 2y = 40$, $x = 40 - 2y$, so

$$A(y) = (40 - 2y)\,y = 40y - 2y^2.$$

Now $A$ depends on the single variable $y$ — ready to differentiate.

### Stating a sensible domain

Lengths are positive, so $y > 0$ and $x = 40 - 2y > 0$, giving $0 < y < 20$. The optimization only searches this **domain**; an answer outside it is rejected.

■ **Reading what the question asks for**

Check whether the question wants the **variable** (the $y$ that optimizes), the **maximum value** ($A$ itself), or both. A common lost mark is giving $y$ when the question asked for the area.

## 2 Practice

Now apply the methods above.

**2.1** A rectangle has perimeter 24 cm. Write its **area** as a function of its width $x$. [2]

**2.2** An open-top box has a square base of side $x$ and volume 32 $\text{cm}^3$. Write its **surface area** (base + 4 sides) as a function of $x$. [2]

**2.3** For the farmer's pen $A(y) = 40y - 2y^2$, state the domain of $y$. [1]

## 3 Exam-style questions

**3.1** A quantity $Q$ is to be maximized subject to a constraint. What is the **first** step? [1]

- **A** differentiate $Q$ immediately
- **B** write $Q$ as a function of a single variable using the constraint
- **C** set $Q = 0$
- **D** find the second derivative

---

**3.2** A cylindrical can has volume 355 cm$^3$. Its surface area is $S = 2\pi r^2 + 2\pi rh$.

(a) Use the volume constraint $\pi r^2 h = 355$ to write $S$ as a function of $r$ only. [2]

(b) State the domain of $r$. [1]

**3.3** A 600 cm$^2$ sheet of card is folded into an open box with a square base of side $x$ and height $h$, using all the card ($x^2 + 4xh = 600$). Write the **volume** $V$ as a function of $x$ alone. [3]

## Solutions

**2.1** Perimeter $2x + 2w = 24 \Rightarrow$ length $= 12 - x$; $A = x(12 - x) = 12x - x^2$.

**2.2** Height from $x^2h = 32$, so $h = 32/x^2$; $S = x^2 + 4x\left(\dfrac{32}{x^2}\right) = x^2 + \dfrac{128}{x}$.

**2.3** $0 < y < 20$.

**3.1** **B** — reduce to a single variable using the constraint before differentiating.

**3.2** (a) $h = \dfrac{355}{\pi r^2}$; $S(r) = 2\pi r^2 + 2\pi r\left(\dfrac{355}{\pi r^2}\right) = 2\pi r^2 + \dfrac{710}{r}$. (b) $r > 0$.

**3.3** $h = \dfrac{600 - x^2}{4x}$; $V = x^2h = x^2\left(\dfrac{600 - x^2}{4x}\right) = \dfrac{600x - x^3}{4} = 150x - \frac{1}{4}x^3$.