Exercise walk-through

Volume with Washer Method: Revolving Around the x- or y-Axis ▪ 8.11

eXercise

You must be able to

- identify the **outer** and **inner** radii of a washer (ring)
- integrate $\pi\left(R_{\text{outer}}^2 - R_{\text{inner}}^2\right)$ (the **washer method 垫圈法**)
- set each radius as a distance from a curve to the axis

Method — When there is a hole

If the region does **not** touch the axis, revolving it leaves a hole, so each slice is a **washer** (a ring). Its area is (outer disc) − (inner disc):

$$V = \pi \int_a^b \left(R_{\text{outer}}^2 - R_{\text{inner}}^2\right) dx.$$

Method — Identifying the radii

Revolve the region between $y = \sqrt{x}$ (top) and $y = x$ (bottom) on $[0, 1]$ about the x-axis. Outer radius (further curve) $R = \sqrt{x}$; inner radius $r = x$:

$$V = \pi \int_0^1 \left((\sqrt{x})^2 - x^2 \right) dx = \pi \int_0^1 (x - x^2)\, dx = \pi \left(\tfrac{1}{2} - \tfrac{1}{3} \right) = \tfrac{\pi}{6}.$$

Method — Do not square then subtract wrongly

$R^2 - r^2$ is **not** $(R - r)^2$. Square each radius separately, then subtract.

Method — Choosing outer vs inner

The **outer** radius comes from the curve **farther** from the axis; the **inner** from the curve **nearer** the axis. A quick sketch settles which is which.

Practice 2.1 ▪ 1 mark

State the washer-method formula for the volume.

Solution 2.1

Worked steps

$$V = \pi \int_a^b \left(R_{\text{outer}}^2 - R_{\text{inner}}^2\right) dx \quad \boxed{\text{B1}}.$$

Practice 2.2 ▪ 1 mark

For $R = 4$ and $r = 2$, find $\pi(R^2 - r^2)$.

Solution 2.2

Worked steps

$\pi(16-4) = 12\pi$ B1.

Practice 2.3 ▪ 2 marks

Evaluate $\pi \int_0^1 (x - x^2)\, dx$.

Solution 2.3

Worked steps

$\pi\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = \pi\left(\frac{1}{2} - \frac{1}{3}\right) = \frac{\pi}{6}$ M1 A1.

Exam-style 3.1 ▪ 1 mark

The washer method integrates

A $\pi \int (R - r)^2 \, dx$

B $\pi \int (R^2 - r^2) \, dx$

C $\pi \int R^2 \, dx$

D $\int (R^2 - r^2) \, dx$

Solution 3.1

A $\pi \int (R - r)^2 \, dx$

B $\pi \int (R^2 - r^2) \, dx$ ✓

C $\pi \int R^2 \, dx$

D $\int (R^2 - r^2) \, dx$

Worked steps

B — subtract the **squares** of the radii.

Exam-style 3.2 ▪ 2 marks

The region between $y = x$ and $y = x^2$ (from $x = 0$ to $x = 1$) is revolved about the x-axis.

(a) Identify the outer and inner radii.

(b) Set up and evaluate the volume.

Solution 3.2

Method: Identifying the radii

Worked steps

(a) On $(0,1)$, $y = x$ is above $y = x^2$; outer $R = x$, inner $r = x^2$ M1 A1. (b) $V = \pi \int_0^1 (x^2 - x^4)\,dx = \pi\left[\frac{x^3}{3} - \frac{x^5}{5}\right]_0^1 = \pi\left(\frac{1}{3} - \frac{1}{5}\right) = \frac{2\pi}{15}$ M1 M1 M1 A1.

Exam-style 3.3 ▪ 3 marks

The region bounded by $y = 4$ and $y = x^2$ (for $0 \leq x \leq 2$) is revolved about the x-axis. Set up the washer integral for the volume.

Solution 3.3

Worked steps

Outer $R = 4$, inner $r = x^2$; $V = \pi \int_0^2 \left(4^2 - (x^2)^2\right) dx = \pi \int_0^2 (16 - x^4)\, dx$ M1 M1 A1.