Past-paper walk-through

Exponential Models with Differential Equations ▪ 7.8

eXercise

Questions in this set

- 2021 Q6

Reading the mark scheme

- **B1** a mark that stands on its own – the point is either made or not.
- **M1** a *method* mark: the right approach, even if the number is wrong.
- **A1** an *answer* mark, and it usually needs its M mark first.
- **C1** a working mark on the way to the answer.
- // separates answers the examiner accepts equally.
- **ecf** = error carried forward: a later part is marked on your own earlier answer.
- **owtte** = "or words to that effect" – the wording need not match.

Question 6

2021 ▪ Q6

A medication is administered to a patient. The amount, in milligrams, of the medication in the patient at time t hours is modeled by a function $y = A(t)$ that satisfies the differential equation $\dfrac{dy}{dt} = \dfrac{12 - y}{3}$. At time $t = 0$ hours, there are 0 milligrams of the medication in the patient.

(a) A portion of the slope field for the differential equation $\dfrac{dy}{dt} = \dfrac{12 - y}{3}$ is given below. Sketch the solution curve through the point $(0, 0)$.
[Slope field diagram, t-axis horizontal from 0 rightward, y-axis vertical; short line segments drawn at grid points showing slope direction: segments near the top of the grid slope downward (negative slope, since $y > 12$ there), a horizontal row of flat dashes at $y = 12$ (zero slope), and segments below slope upward (positive slope, since

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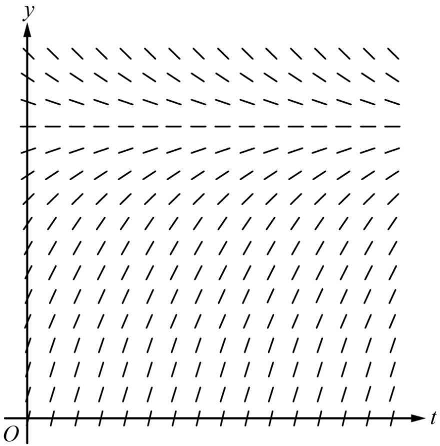
$y < 12$), with segments closer to $y = 0$ sloping steeply upward and flattening as y approaches 12 across each column.]

Question 6



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(b) Using correct units, interpret the statement $\lim_{t \to \infty} A(t) = 12$ in the context of this problem.

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(c) Use separation of variables to find $y = A(t)$, the particular solution to the differential equation $\dfrac{dy}{dt} = \dfrac{12 - y}{3}$ with initial condition $A(0) = 0$.

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(d) A different procedure is used to administer the medication to a second patient. The amount, in milligrams, of the medication in the second patient at time t hours is modeled by a function $y = B(t)$ that satisfies the differential equation $\dfrac{dy}{dt} = 3 - \dfrac{y}{t+2}$. At time $t = 1$ hour, there are 2.5 milligrams of the medication in the second patient. Is the rate of change of the amount of medication in the second patient increasing or decreasing at time $t = 1$? Give a reason for your answer.

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(a) Mark scheme

- Sketch the solution curve to $\dfrac{dy}{dt} = \dfrac{12 - y}{3}$ through $(0, 0)$: the curve must start at $(0, 0)$, be generally increasing, be concave down throughout, and approach the horizontal asymptote $y = 12$ from below as t increases (following the slope-field segments, which are positive below $y = 12$, flat at $y = 12$, and negative above $y = 12$). Point 1: earned only if a single such curve is drawn passing through $(0, 0)$ with the correct increasing/concave-down/asymptotic shape; not earned if two or more solution curves are presented.

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(b)

Mark scheme

- Over time, the amount of medication in the patient approaches 12 milligrams. Point 1: the interpretation must include "medication in the patient," "approaches 12," and units (milligrams), or their equivalents.

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(c) Mark scheme

- Separate variables: $\dfrac{dy}{dt} = \dfrac{12-y}{3} \Rightarrow \dfrac{dy}{12-y} = \dfrac{dt}{3}$. Integrate: $\displaystyle\int \frac{dy}{12-y} = \int \frac{dt}{3} \Rightarrow -\ln|12-y| = \frac{t}{3} + C$. Solve for y: $\ln|12-y| = -\dfrac{t}{3} - C \Rightarrow |12-y| = e^{-t/3-C} \Rightarrow y = 12 + Ke^{-t/3}$. Apply $A(0) = 0$: $0 = 12 + K \Rightarrow K = -12$. Final: $y = A(t) = 12 - 12e^{-t/3}$. Point 1: correct separation of variables ($dy/(12-y) = dt/3$; a "bad separation" like $dy/(12-y) = 3\,dt$ does not earn this point but the response stays eligible for points 2-3). Point 2: correct antiderivatives on both sides (absolute value bars not required). Point 3: correctly introduces and solves for the constant of integration

Solution 6

using the initial condition $A(0) = 0$ (earned consecutively after point 2). Point 4: solves explicitly for $y = A(t) = 12 - 12e^{-t/3}$ (needs points 1-3 in the standard path).

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(d) Mark scheme

- $\dfrac{dy}{dt} = 3 - \dfrac{y}{t+2} \Rightarrow \dfrac{d^2y}{dt^2} = -\dfrac{\frac{dy}{dt}(t+2) - y}{(t+2)^2}$ (quotient rule). At $t = 1$: $B'(1) = 3 - \dfrac{B(1)}{3} = 3 - \dfrac{2.5}{3} = \dfrac{6.5}{3}$. Then $B''(1) = -\dfrac{B'(1)\cdot 3 - B(1)}{3^2} = -\dfrac{6.5 - 2.5}{9} = -\dfrac{4}{9} < 0$. The rate of change of the amount of medication in the second patient is DECREASING at $t = 1$ because $B''(1) < 0$ and d^2y/dt^2 is continuous on an interval containing $t = 1$. Point 1: correctly applies the quotient rule to $y/(t+2)$ (or product rule to $y(t+2)^{-1}$); errors differentiating the constant 3 or mishandling the sign forfeit this point. Point 2: states $B''(1) < 0$ with a correct value/sign — cannot be earned unless

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the second derivative itself is correct. Point 3: consistent conclusion (decreasing) based on the declared value/sign of $B''(1)$; only needs an attempt at the quotient/product rule to be eligible.