Past-paper walk-through

Finding General Solutions Using Separation of Variables ▪ 7.6

eXercise

Questions in this set

- 2024 Q3
- 2023 Q3
- 2022 Q5
- 2017 Q4
- 2014 Q6

Reading the mark scheme

- **B1** a mark that stands on its own – the point is either made or not.
- **M1** a *method* mark: the right approach, even if the number is wrong.
- **A1** an *answer* mark, and it usually needs its M mark first.
- **C1** a working mark on the way to the answer.
- // separates answers the examiner accepts equally.
- **ecf** = error carried forward: a later part is marked on your own earlier answer.
- **owtte** = "or words to that effect" – the wording need not match.

Question 3

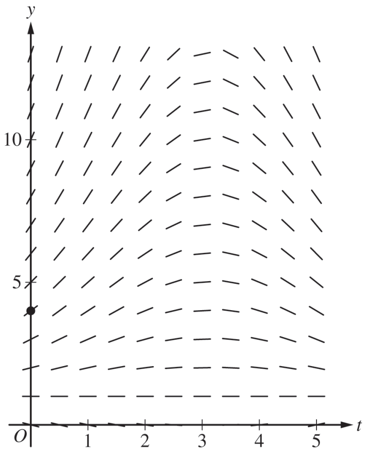
2024 ▪ Q3

The depth of seawater at a location can be modeled by the function H that satisfies the differential equation $\dfrac{dH}{dt} = \dfrac{1}{2}(H - 1)\cos\left(\dfrac{t}{2}\right)$, where $H(t)$ is measured in feet and t is measured in hours after noon $(t = 0)$. It is known that $H(0) = 4$.

(a) A portion of the slope field for the differential equation is provided. Sketch the solution curve, $y = H(t)$, through the point $(0, 4)$.

[Slope field graph; t-axis (horizontal) labeled 1 to 5, y-axis (vertical) labeled 5 and 10; a dot is plotted at $(0, 4)$; short line segments show the slope at each grid point — segments are steep and negatively-sloped (like "") in the upper-right region, nearly flat in the lower region, and forward-sloped (like "/") in the upper-left region, consistent with the given differential equation.]

Question 3



Question 3

2024 ▪ Q3

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(b) For $0 < t < 5$, it can be shown that $H(t) > 1$. Find the value of t, for $0 < t < 5$, at which H has a critical point. Determine whether the critical point corresponds to a relative minimum, a relative maximum, or neither a relative minimum nor a relative maximum of the depth of seawater at the location. Justify your answer.

Question 3

2024 ▪ Q3

The depth of seawater at a location can be modeled by the function H that satisfies the differential equation $\frac{dH}{dt} = \frac{1}{2}(H-1)\cos\left(\frac{t}{2}\right)$, where $H(t)$ is measured in feet and t is measured in hours after noon $(t=0)$. It is known that $H(0) = 4$.

(c) Use separation of variables to find $y = H(t)$, the particular solution to the differential equation $\frac{dH}{dt} = \frac{1}{2}(H-1)\cos\left(\frac{t}{2}\right)$ with initial condition $H(0) = 4$.

Solution 3

(a) Mark scheme

- The solution curve must pass through $(0, 4)$ and follow the given slope field: it rises (following the steep positive-then-shallowing slopes), reaches a relative maximum near $t = \pi \approx 3.14$ (height about 9.4, matching part (b)'s critical point), then decreases for $\pi < t < 5$, staying consistent with the marked segments and extending to at least $t = 4.5$ with no obvious conflicts with the drawn slope lines. 1 point awarded only if the curve passes through $(0, 4)$, extends to at least $t = 4.5$, and has no obvious conflicts with the given slope lines (only the portion inside the given field is considered).

Solution 3

(b) Mark scheme

- Since $H(t) > 1$ on $0 < t < 5$, $\dfrac{dH}{dt} = 0$ requires $\cos\left(\dfrac{t}{2}\right) = 0$, which gives $t = \pi$. For $0 < t < \pi$, $\dfrac{dH}{dt} > 0$, and for $\pi < t < 5$, $\dfrac{dH}{dt} < 0$; therefore $t = \pi$ is the location of a relative maximum of H (the depth of seawater). 1 point for considering the sign of dH/dt (or equivalently $\cos(t/2)$), 1 point for identifying $t = \pi$ as the critical point, 1 point (only earnable with the first) for the correct classification 'relative maximum' with justification via a sign analysis of dH/dt on either side of $t = \pi$ (or an equivalent Second Derivative Test).

Solution 3

(c) Mark scheme

- Separate variables: $\dfrac{dH}{H-1} = \dfrac{1}{2}\cos\left(\dfrac{t}{2}\right)\,dt$. Integrate both sides: $\ln|H-1| = \sin\left(\dfrac{t}{2}\right) + C$. Apply $H(0) = 4$: $\ln|4-1| = \sin(0) + C \Rightarrow C = \ln 3$; since $H(0) = 4 > 1$, $|H-1| = H-1$, so $\ln(H-1) = \sin(t/2) + \ln 3$. Solve for H: $H - 1 = e^{\sin(t/2)+\ln 3} = 3e^{\sin(t/2)}$, so $H(t) = 1 + 3e^{\sin(t/2)}$. 1 point for correct separation of variables, 1 point for one correct antiderivative (either side), 1 point for the second correct antiderivative, 1 point for including the constant of integration and correctly using the initial condition $H(0) = 4$ (eligible only if the separation point and at least one antiderivative point were earned), 1 point for the

Solution 3

fully solved final answer $H(t) = 1 + 3e^{\sin(t/2)}$ (eligible only if the first four points were earned; a response with no separation of variables earns 0 of 5).

Question 3

2023 ▪ Q3

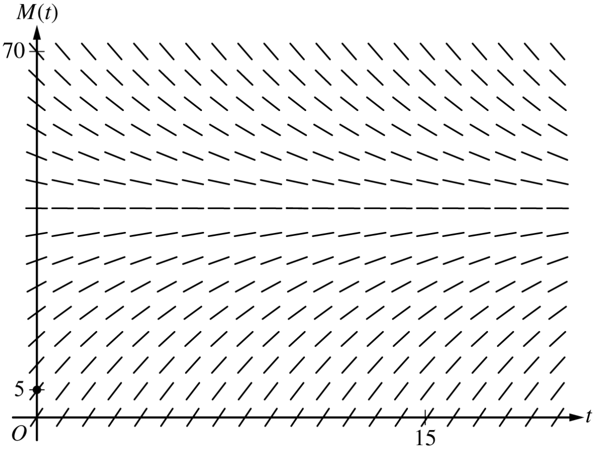
A bottle of milk is taken out of a refrigerator and placed in a pan of hot water to be warmed. The increasing function M models the temperature of the milk at time t, where $M(t)$ is measured in degrees Celsius ($^\circ$C) and t is the number of minutes since the bottle was placed in the pan. M satisfies the differential equation $\dfrac{dM}{dt} = \dfrac{1}{4}(40 - M)$. At time $t = 0$, the temperature of the milk is 5°C. It can be shown that $M(t) < 40$ for all values of t.

(a) A slope field for the differential equation $\dfrac{dM}{dt} = \dfrac{1}{4}(40 - M)$ is shown. Sketch the solution curve through the point $(0, 5)$.
[Slope field diagram: axes labeled $M(t)$ (vertical, with gridlines at 5 and 70 marked) and t (horizontal, with a gridline at 15 marked), origin O. Short line segments are

Question 3

drawn at grid points across the region $0 \le t \le 15$-ish, $0 \le M(t) \le 70$-ish: segments are nearly horizontal (flat, slightly downward-sloping) near $M(t) = 70$, becoming steeper negative slope moving down toward $M(t) \approx 40$ (segments are horizontal/flat right at $M(t) = 40$), then the slopes become positive and increasingly steep as $M(t)$ decreases from 40 down toward $M(t) = 5$, with the steepest (near-vertical, slope ≈ 1) segments at the bottom of the field near $M(t) = 5$. The point $(0, 5)$ is marked with a star on the vertical axis.]

Question 3



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2023 ▪ Q3

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(b) Use the line tangent to the graph of M at $t = 0$ to approximate $M(2)$, the temperature of the milk at time $t = 2$ minutes.

Question 3

2023 ▪ Q3

A bottle of milk is taken out of a refrigerator and placed in a pan of hot water to be warmed. The increasing function M models the temperature of the milk at time t, where $M(t)$ is measured in degrees Celsius $(^{\circ}\mathrm{C})$ and t is the number of minutes since the bottle was placed in the pan. M satisfies the differential equation $\frac{dM}{dt} = \frac{1}{4}(40 - M)$. At time $t = 0$, the temperature of the milk is $5^{\circ}\mathrm{C}$. It can be shown that $M(t) < 40$ for all values of t.

(c) Write an expression for $\frac{d^2M}{dt^2}$ in terms of M. Use $\frac{d^2M}{dt^2}$ to determine whether the approximation from part (b) is an underestimate or an overestimate for the actual value of $M(2)$. Give a reason for your answer.

Question 3

2023 ▪ Q3

A bottle of milk is taken out of a refrigerator and placed in a pan of hot water to be warmed. The increasing function M models the temperature of the milk at time t, where $M(t)$ is measured in degrees Celsius ($^\circ$C) and t is the number of minutes since the bottle was placed in the pan. M satisfies the differential equation $\frac{dM}{dt} = \frac{1}{4}(40 - M)$. At time $t = 0$, the temperature of the milk is 5°C. It can be shown that $M(t) < 40$ for all values of t.

(d) Use separation of variables to find an expression for $M(t)$, the particular solution to the differential equation $\frac{dM}{dt} = \frac{1}{4}(40 - M)$ with initial condition $M(0) = 5$.

Solution 3

(a)

Mark scheme

- The solution curve through (0,5) starts at (0,5), increases and is concave down, and approaches (curving up toward, but always staying below) the flat slope segments at M=40 as t increases across the shown window; it must not cross any drawn slope-field segments (1 pt: curve passes through (0,5), extends reasonably close to the left and right edges of the given rectangle, has no obvious conflicts with the slope lines, and lies entirely below M=40).

Solution 3

(b) Mark scheme

- $\left.\frac{dM}{dt}\right|_{t=0} = \frac{1}{4}(40-5) = \frac{35}{4}$ (1 pt: value of the derivative at t=0). Tangent line: $y = 5 + \frac{35}{4}(t-0)$, so $M(2) \approx 5 + \frac{35}{4} \cdot 2 = 22.5$; the temperature of the milk at t=2 minutes is approximately 22.5°C (1 pt: approximation using a tangent line through (0,5) with slope 35/4, correctly simplified).

Solution 3

(c) Mark scheme

- $\frac{d^2M}{dt^2} = -\frac{1}{4}\frac{dM}{dt} = -\frac{1}{4}\left(\frac{1}{4}(40 - M)\right) = -\frac{1}{16}(40 - M)$ (1 pt: correct second-derivative expression in terms of M, no subsequent simplification error). Since $M(t) < 40$, $\frac{d^2M}{dt^2} < 0$, so the graph of M is concave down; therefore the tangent-line approximation from part (b) is an overestimate (1 pt: overestimate conclusion with a reason based on concavity/decreasing dM/dt).

Solution 3

(d) Mark scheme

- Separate variables: $\dfrac{dM}{40-M} = \dfrac{1}{4}\,dt$ (1 pt: separates variables). Integrate: $\displaystyle\int \frac{dM}{40-M} = \int \frac{1}{4}\,dt \Rightarrow -\ln|40-M| = \frac{1}{4}t + C$ (1 pt: finds antiderivatives). Apply the initial condition M(0)=5: $-\ln|40-5| = 0 + C \Rightarrow C = -\ln 35$; since M(t)<40, $|40-M| = 40-M$, giving $-\ln(40-M) = \dfrac{1}{4}t - \ln 35$, i.e. $\ln(40-M) = -\dfrac{1}{4}t + \ln 35$ (1 pt: constant of integration correctly found and initial condition used). Solve for M: $40 - M = 35e^{-t/4} \Rightarrow M(t) = 40 - 35e^{-t/4}$ (1 pt: solves for M — the final answer must be exactly this closed form, or equivalent).

Question 5

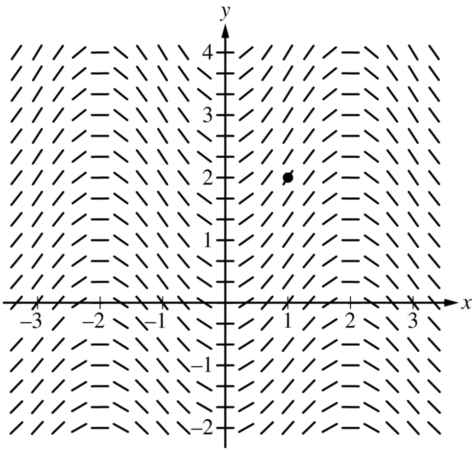
2022 ▪ Q5

Consider the differential equation $\dfrac{dy}{dx} = \dfrac{1}{2}\sin\left(\dfrac{\pi}{2}x\right)\sqrt{y+7}$. Let $y = f(x)$ be the particular solution to the differential equation with the initial condition $f(1) = 2$. The function f is defined for all real numbers.

(a) A portion of the slope field for the differential equation is given below. Sketch the solution curve through the point $(1, 2)$.

[Graph: a slope field on the region $-3 \le x \le 3$, $-2 \le y \le 4$ (integer gridlines on both axes). Short line segments at each grid point show the local slope; segments in the left portion (roughly $x < 0$) slant like "/" (positive slope, forward slashes), and segments in the right portion (roughly $x > 0$) slant like "(negative slope, back slashes), with a marked point at $(1, 2)$ where the solution curve should be sketched.]

Question 5



Question 5

2022 ▪ Q5

Consider the differential equation $\frac{dy}{dx} = \frac{1}{2}\sin\left(\frac{\pi}{2}x\right)\sqrt{y+7}$. Let $y = f(x)$ be the particular solution to the differential equation with the initial condition $f(1) = 2$. The function f is defined for all real numbers.

(b) Write an equation for the line tangent to the solution curve in part (a) at the point $(1, 2)$. Use the equation to approximate $f(0.8)$.

Question 5

2022 ▪ Q5

Consider the differential equation $\frac{dy}{dx} = \frac{1}{2}\sin\left(\frac{\pi}{2}x\right)\sqrt{y+7}$. Let $y = f(x)$ be the particular solution to the differential equation with the initial condition $f(1) = 2$. The function f is defined for all real numbers.

(c) It is known that $f''(x) > 0$ for $-1 \leq x \leq 1$. Is the approximation found in part (b) an overestimate or an underestimate for $f(0.8)$? Give a reason for your answer.

Question 5

2022 ▪ Q5

Consider the differential equation $\frac{dy}{dx} = \frac{1}{2}\sin\left(\frac{\pi}{2}x\right)\sqrt{y+7}$. Let $y = f(x)$ be the particular solution to the differential equation with the initial condition $f(1) = 2$. The function f is defined for all real numbers.

(d) Use separation of variables to find $y = f(x)$, the particular solution to the differential equation

$$\frac{dy}{dx} = \frac{1}{2}\sin\left(\frac{\pi}{2}x\right)\sqrt{y+7}$$

with the initial condition $f(1) = 2$.

Solution 5

(a)

Mark scheme

- The solution curve must pass through $(1, 2)$, follow the given slope field, satisfy $f(x) > 0$ at every point on the curve, extend reasonably close to the left and right edges of the given square, and have any local maximum/minimum occur only where the slope field shows horizontal segments. This is a graphical sketch, not an algebraic answer; 1 point is earned for a curve satisfying these properties.

Solution 5

(b)

Mark scheme

- $\left.\dfrac{dy}{dx}\right|_{(x,y)=(1,2)} = \dfrac{1}{2}\sin\left(\dfrac{\pi}{2}\right)\sqrt{2+7} = \dfrac{1}{2}\cdot 1\cdot 3 = \dfrac{3}{2}$. Tangent line: $y = 2 + \dfrac{3}{2}(x-1)$. Approximation: $f(0.8) \approx 2 + \dfrac{3}{2}(0.8-1) = 1.7$. Points: (1) correct tangent line equation (any equivalent form); (2) correct approximation $f(0.8) \approx 1.7$.

Solution 5

(c)

Mark scheme

- Because $f''(x) > 0$ on $-1 \leq x \leq 1$, f is concave up there, so the tangent line lies below the graph of $y = f(x)$ at $x = 0.8$, and the approximation for $f(0.8)$ found in part (b) is an underestimate. 1 point for the correct answer with a reason that references $f''(x) > 0$, f' increasing, or f concave up.

Solution 5

(d) Mark scheme

- $\dfrac{dy}{dx} = \dfrac{1}{2}\sin\left(\dfrac{\pi}{2}x\right)\sqrt{y+7} \Rightarrow \displaystyle\int \frac{dy}{\sqrt{y+7}} = \int \frac{1}{2}\sin\left(\frac{\pi}{2}x\right)\,dx$. Integrating: $2\sqrt{y+7} = -\dfrac{1}{\pi}\cos\left(\dfrac{\pi}{2}x\right) + C$. Using $f(1) = 2$: $2\sqrt{2+7} = -\dfrac{1}{\pi}\cos\left(\dfrac{\pi}{2}\right) + C \Rightarrow 6 = 0 + C \Rightarrow C = 6$. So $\sqrt{y+7} = 3 - \dfrac{1}{2\pi}\cos\left(\dfrac{\pi}{2}x\right)$, and $y = \left(3 - \dfrac{1}{2\pi}\cos\left(\dfrac{\pi}{2}x\right)\right)^2 - 7$. Points: (1) correct separation of variables; (2) one correct antiderivative; (3) the other correct antiderivative; (4) correct constant of integration found using the initial condition; (5) correctly solves for y.

Question 4

2017 ▪ Q4

At time $t = 0$, a boiled potato is taken from a pot on a stove and left to cool in a kitchen. The internal temperature of the potato is 91 degrees Celsius ($^\circ C$) at time $t = 0$, and the internal temperature of the potato is greater than $27^\circ C$ for all times $t > 0$. The internal temperature of the potato at time t minutes can be modeled by the function H that satisfies the differential equation $\dfrac{dH}{dt} = -\dfrac{1}{4}(H - 27)$, where $H(t)$ is measured in degrees Celsius and $H(0) = 91$.

(a) Write an equation for the line tangent to the graph of H at $t = 0$. Use this equation to approximate the internal temperature of the potato at time $t = 3$.

(b) Use $\dfrac{d^2H}{dt^2}$ to determine whether your answer in part (a) is an underestimate or an overestimate of the internal temperature of the potato at time $t = 3$.

Question 4

(c) For $t < 10$, an alternate model for the internal temperature of the potato at time t minutes is the function G that satisfies the differential equation $\dfrac{dG}{dt} = -(G - 27)^{2/3}$, where $G(t)$ is measured in degrees Celsius and $G(0) = 91$. Find an expression for $G(t)$. Based on this model, what is the internal temperature of the potato at time $t = 3$?

Question 6

2014 ▪ Q6

Consider the differential equation $\frac{dy}{dx} = (3 - y)\cos x$. Let $y = f(x)$ be the particular solution to the differential equation with the initial condition $f(0) = 1$. The function f is defined for all real numbers.

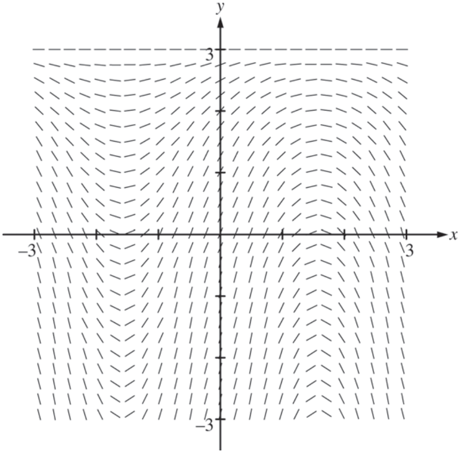
(a) A portion of the slope field of the differential equation is given below. Sketch the solution curve through the point $(0, 1)$.

[Slope field diagram on the domain $-3 \leq x \leq 3$, $-3 \leq y \leq 3$, with tick marks at -3, 1, 2(implied), 3 on both axes (visible ticks at -3 and 3 on the x-axis, at 3 and -3 on the y-axis, plus unlabelled tick marks at intermediate integer values). Short line segments show the slope at each grid point: segments are roughly horizontal (dashes) in the upper-left and upper-right regions (where y is large, slope near 0 since $3 - y$ is very negative but cos x oscillates - shown as near-flat dashes), segments tilt more

Question 6

steeply (steep near-vertical strokes) in the band around $y \approx 1$ to 2 near $x = 0$, and the pattern is symmetric in a banded way across the vertical line $x = 0$, repeating roughly periodically along the x-axis consistent with the $\cos x$ factor.]

Question 6



Question 6

2014 ▪ Q6

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(b) Write an equation for the line tangent to the solution curve in part (a) at the point $(0, 1)$. Use the equation to approximate $f(0.2)$.

Question 6

2014 ▪ Q6

Consider the differential equation $\frac{dy}{dx} = (3 - y)\cos x$. Let $y = f(x)$ be the particular solution to the differential equation with the initial condition $f(0) = 1$. The function f is defined for all real numbers.

(c) Find $y = f(x)$, the particular solution to the differential equation with the initial condition $f(0) = 1$.

Solution 6

(a) Mark scheme

- Sketch the solution curve through (0,1), following the slope-field segment directions: the curve should dip to a local minimum to the left of x=0 (matching the field's near-horizontal/negative segments there) and rise through (0,1) toward a maximum further to the right, staying consistent with the plotted slope segments across -3$\leq x \leq 3(\mathit{matchestheexactsolutiony} = 3 - 2e^{-\sin x}, \mathit{whichhasaminimumnearx} = -\pi/2\mathit{andamaximumnearx} = \pi/2).\mathit{Points}: 1\mathit{foracurvethatpassesthrough}(0,1)\mathit{andisconsistentwiththegivenslopefield}'\mathit{ssegmentdirectionsthr}$

Solution 6

(b)

Mark scheme

- $\frac{dy}{dx}\Big|_{(0,1)} = (3-1)\cos(0) = 2\cdot 1 = 2. Tangentline: y = 2x+1 (point-slopethrough (0,1) withslope 2). Usingthetangentline, f(0.2$

Solution 6

(c) Mark scheme

- Separate variables:

dy $\overline{dx{=}(3{-}y)\cos x{\Rightarrow}\int\frac{dy}{3-y}{=}\int\cos x\,dx{\Rightarrow}-\ln|3{-}y|{=}\sin x{+}C. Applytheinitialconditionf(0){=}1{:}-\ln|3{-}1|{=}\sin(0){+}C{\Rightarrow}-\ln}$