Past-paper walk-through

The Fundamental Theorem of Calculus and Definite Integrals ▪ 6.7

eXercise

Questions in this set

- 2026 Q6
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- 2016 Q6
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Reading the mark scheme

- **B1** a mark that stands on its own – the point is either made or not.
- **M1** a *method* mark: the right approach, even if the number is wrong.
- **A1** an *answer* mark, and it usually needs its M mark first.
- **C1** a working mark on the way to the answer.
- // separates answers the examiner accepts equally.
- **ecf** = error carried forward: a later part is marked on your own earlier answer.
- **owtte** = "or words to that effect" – the wording need not match.

Question 6

2026 ▪ Q6

The function f is twice differentiable. The table gives values of f and its derivative f' at selected values of x.

x	**0**	**2**	**3**	**6**
$f(x)$	-1	3	8	5
$f'(x)$	-5	4	9	-2

Question 6

(a) Find $\lim_{x\to 2} \frac{f(x)}{x}$, or state that the limit does not exist.

(b) Let $g(x) = f(f(x))$. Find $g'(2)$. Show the work that leads to your answer.

(c) Let h be a differentiable function such that $h(0) = 10$ and $h'(x) = f'(3x)$. Find $h(2)$. Show the work that leads to your answer.

(d)(i) Let k be the function defined by $k(x) = \int_0^x t^2 f(t)\, dt$. Find $k'(x)$.

(d)(ii) Find $k''(3)$. Show the work that leads to your answer.

Question 6

x	0	2	3	6
$f(x)$	-1	3	8	5
$f'(x)$	-5	4	9	-2

Question 3

2025 ▪ Q3

A student starts reading a book at time $t = 0$ minutes and continues reading for the next 10 minutes. The rate at which the student reads is modeled by the differentiable function R, where $R(t)$ is measured in words per minute. Selected values of $R(t)$ are given in the table shown.

t **(minutes)**	**0**	**2**	**8**	**10**
$R(t)$ (words per minute)	90	100	150	162

Question 3

(a) Approximate $R'(1)$ using the average rate of change of R over the interval $0 \le t \le 2$. Show the work that leads to your answer. Indicate units of measure.
(b) Must there be a value c, for $0 < c < 10$, such that $R(c) = 155$? Justify your answer.
(c) Use a trapezoidal sum with the three subintervals indicated by the data in the table to approximate the value of $\int_0^{10} R(t)\,dt$. Show the work that leads to your answer.
(d) A teacher also starts reading at time $t = 0$ minutes and continues reading for the next 10 minutes. The rate at which the teacher reads is modeled by the function W defined by $W(t) = -\frac{3}{10}t^2 + 8t + 100$, where $W(t)$ is measured in words per minute. Based on the model, how many words has the teacher read by the end of the 10 minutes? Show the work that leads to your answer.

Question 3

t (minutes)	0	2	8	10
$R(t)$ (words per minute)	90	100	150	162

A. Approximate $R'(1)$ using the average rate of change of R over the interval $0 \le t \le 2$. Show the work that leads to your answer. Indicate units of measure.

B. Must there be a value c, for $0 < c < 10$, such that $R(c) = 155$? Justify your answer.

C. Use a trapezoidal sum with the three subintervals indicated by the data in the table to approximate the value of $\int_0^{10} R(t)\,dt$. Show the work that leads to your answer.

D. A teacher also starts reading at time $t = 0$ minutes and continues reading for the next 10

Solution 3

(a) Mark scheme

- $R'(1) \approx \dfrac{R(2)-R(0)}{2-0} = \dfrac{100-90}{2} = \dfrac{10}{2} = 5$ words per minute per minute. Point 1: correct answer with the supporting difference-and-quotient setup using table values (e.g. $\dfrac{R(2)-R(0)}{2-0} = 5$; the quotient alone without the difference/values is not sufficient). Point 2: correct units, words per minute per minute (also accepted: words/minute^2).

Solution 3

(b)

Mark scheme

- R is differentiable, so R is continuous. $R(0) = 90 < 155 < R(10) = 162$. By the Intermediate Value Theorem, there must be a value c, with $0 < c < 10$, such that $R(c) = 155$. Point 3: states R is continuous because R is differentiable (with justification, not just asserted). Point 4: correct "yes" answer with justification citing $R(0) < 155 < R(10)$ (or $R(2) < 155$ or $R(8) < 155$ in place of $R(0)$) and continuity (the Intermediate Value Theorem need not be named explicitly, but if named must be correct).

Solution 3

(c) Mark scheme

- Trapezoidal sum over $[0,2],[2,8],[8,10]$: $\int_0^{10} R(t)\,dt \approx \frac{R(0)+R(2)}{2}(2-0)+\frac{R(2)+R(8)}{2}(8-2)+\frac{R(8)+R(10)}{2}(10-8) = \frac{90+100}{2}(2)+\frac{100+150}{2}(6)+\frac{150+162}{2}(2) = 190+750+312 = 1252.$ Point 5: correct form of a trapezoidal sum (three terms, each a product of two factors, with $\frac{1}{2}$ as part of the product; at least 5 of the 6 factors correct). Point 6: correct answer 1252; requires Point 5.

Solution 3

(d) Mark scheme

- $\int_0^{10} W(t)\,dt = \int_0^{10} \left(-\frac{3}{10}t^2 + 8t + 100\right) dt = \left[-\frac{1}{10}t^3 + 4t^2 + 100t\right]_0^{10} =$ $(-100 + 400 + 1000) - 0 = 1300$ words. Point 7: presents the integral of $W(t)$ (indefinite or definite), with or without dt. Point 8: correct antiderivative $-\frac{1}{10}t^3 + 4t^2 + 100t$, with or without the constant of integration. Point 9: correct answer 1300 words; requires Point 8.

Question 4

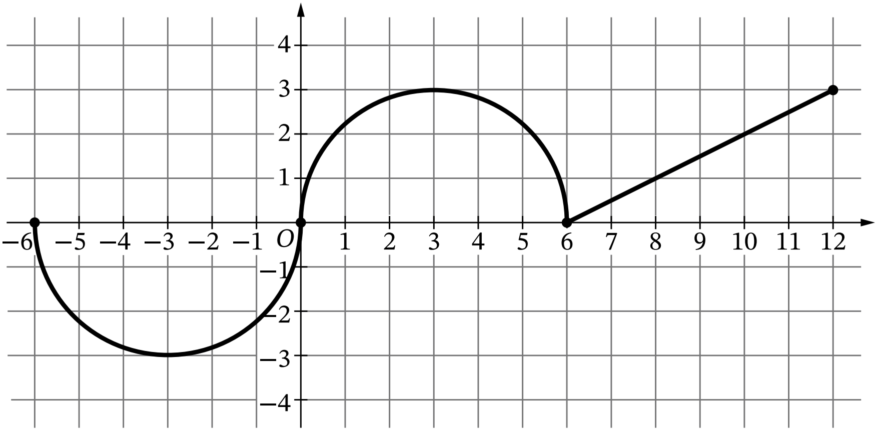
2025 ▪ Q4

The continuous function f is defined on the closed interval $-6 \leq x \leq 12$. The graph of f, consisting of two semicircles and one line segment, is shown in the figure.
[Graph of f: On the interval $-6 \leq x \leq 0$, the graph is a semicircle below the x-axis with endpoints at $(-6, 0)$ and $(0, 0)$, reaching a minimum near $(-3, -3)$. On the interval $0 \leq x \leq 6$, the graph is a semicircle above the x-axis with endpoints at $(0, 0)$ and $(6, 0)$, reaching a maximum near $(3, 3)$. On the interval $6 \leq x \leq 12$, the graph is a line segment from $(6, 0)$ to $(12, 3)$. The horizontal axis is labeled from -6 to 12 and the vertical axis from -4 to 4.]
Graph of f
Let g be the function defined by $g(x) = \int_6^x f(t)\,dt$.
(a) Find $g'(8)$. Give a reason for your answer.

Question 4

(b) Find all values of x in the open interval $-6 < x < 12$ at which the graph of g has a point of inflection. Give a reason for your answer.
(c) Find $g(12)$ and $g(0)$. Label your answers.
(d) Find the value of x at which g attains an absolute minimum on the closed interval $-6 \leq x \leq 12$. Justify your answer.

Question 4



Graph of f

Solution 4

(a)

Mark scheme

- By the Fundamental Theorem of Calculus, $g'(x) = f(x)$, so $g'(8) = f(8) = 1$ (read from the graph of f at $x = 8$, which is on the line segment). Point 1: considers $g'(x) = f(x)$ (or $g' = f$, or $g'(8) = f(8)$). Point 2: correct answer $g'(8) = 1$ (an implied application of the Fundamental Theorem of Calculus, e.g. just "$g'(8) = 1$" or "$f(8) = 1$", is also sufficient for this point alone).

Solution 4

(b) Mark scheme

- The graph of g has a point of inflection where $g'' = f'$ changes sign, i.e. where f changes from decreasing to increasing or vice versa (equivalently, where f attains a relative extremum, or where the slope of f changes sign). From the graph, this happens at $x = -3$, $x = 3$, and $x = 6$. Point 3: correct answer of exactly $x = -3$, $x = 3$, and $x = 6$ (if any other/additional values are declared, neither this point nor the reason point is earned; consideration of $x = -6$ or $x = 12$ does not impact scoring). Point 4: reason correctly tied to the given graph of f (e.g. "f changes from increasing to decreasing or decreasing to increasing there," or "f attains relative extrema there"; a reason relying on an ambiguous term like "the function" or "the graph" does not earn this point).

Solution 4

(c) Mark scheme

- $g(12) = \int_6^{12} f(t)\,dt = \frac{1}{2}(6)(3) = 9$ (area of the triangular line-segment region). $g(0) = \int_6^{0} f(x)\,dx = -\int_0^{6} f(x)\,dx = -\frac{\pi}{2}(3)^2 = -\frac{9\pi}{2}$ (negative of the semicircle area, since integrating from 6 down to 0). Point 5: $g(12) = 9$, with or without supporting work, correctly labeled as $g(12)$ (an unlabeled value does not earn the point). Point 6: $g(0) = -\frac{9\pi}{2}$, with or without supporting work, correctly labeled as $g(0)$.

Solution 4

(d) Mark scheme

- g attains a minimum on $[-6, 12]$ either where $g'(x) = f(x) = 0$ or at an endpoint: candidate points are $x = -6, 0, 6, 12$. Candidates test: $g(-6) = 0$, $g(0) = -\dfrac{9\pi}{2} \approx -14.137$, $g(6) = 0$, $g(12) = 9$. The absolute minimum value is attained at $x = 0$. Point 7: considers $g'(x) = 0$ or $f(x) = 0$ (not earned by just presenting $x = 0$ and $x = 6$ with no justification). Point 8: justification via a correct global (candidates-test) argument evaluating g at all four candidate points $-6, 0, 6, 12$ (values for $g(0)$ and $g(12)$ may be imported from part C); alternatively, an argument that $g'(x) \leq 0$ (or $f(x) \leq 0$) for $-6 \leq x < 0$ and $g'(x) \geq 0$ (or $f(x) \geq 0$) for $0 < x \leq 12$. Point 9: correct answer $x = 0$; requires

Solution 4

Point 8 (a local-argument response, e.g. First/Second Derivative Test, or an incorrect global argument, does not earn Point 8 but remains eligible for Point 9 with the correct answer).

Question 1

2024 ▪ Q1

The temperature of coffee in a cup at time t minutes is modeled by a decreasing differentiable function C, where $C(t)$ is measured in degrees Celsius. For $0 \leq t \leq 12$, selected values of $C(t)$ are given in the table shown.

t (minutes)	0	3	7	12
$C(t)$ (degrees Celsius)	100	85	69	55

(a) Approximate $C'(5)$ using the average rate of change of C over the interval $3 \leq t \leq 7$. Show the work that leads to your answer and include units of measure.

Question 1

t (minutes)	0	3	7	12
$C(t)$ (degrees Celsius)	100	85	69	55

Question 1

2024 ▪ Q1

The temperature of coffee in a cup at time t minutes is modeled by a decreasing differentiable function C, where $C(t)$ is measured in degrees Celsius. For $0 \leq t \leq 12$, selected values of $C(t)$ are given in the table shown.

t (minutes)	0	3	7	12
$C(t)$ (degrees Celsius)	100	85	69	55

(b) Use a left Riemann sum with the three subintervals indicated by the data in the table to approximate the value of $\int_0^{12} C(t)\,dt$. Interpret the meaning of $\dfrac{1}{12}\int_0^{12} C(t)\,dt$ in the context of the problem.

Question 1

2024 ▪ Q1

The temperature of coffee in a cup at time t minutes is modeled by a decreasing differentiable function C, where $C(t)$ is measured in degrees Celsius. For $0 \le t \le 12$, selected values of $C(t)$ are given in the table shown.

t (minutes)	0	3	7	12
$C(t)$ (degrees Celsius)	100	85	69	55

(c) For $12 \le t \le 20$, the rate of change of the temperature of the coffee is modeled by $C'(t) = \dfrac{-24.55e^{0.01t}}{t}$, where $C'(t)$ is measured in degrees Celsius per minute. Find the temperature of the coffee at time $t = 20$. Show the setup for your calculations.

Question 1

2024 ▪ Q1

The temperature of coffee in a cup at time t minutes is modeled by a decreasing differentiable function C, where $C(t)$ is measured in degrees Celsius. For $0 \leq t \leq 12$, selected values of $C(t)$ are given in the table shown.

t (minutes)	0	3	7	12
$C(t)$ (degrees Celsius)	100	85	69	55

(d) For the model defined in part (c), it can be shown that $C''(t) = \dfrac{0.2455e^{0.01t}(100 - t)}{t^2}$. For $12 < t < 20$, determine whether the temperature of the coffee is changing at a decreasing rate or at an increasing rate. Give a reason for your answer.

Solution 1

(a)

Mark scheme

- $C'(5) \approx \dfrac{C(7) - C(3)}{7 - 3} = \dfrac{69 - 85}{4} = -4$ degrees Celsius per minute. 1 point for the difference-quotient setup with supporting work (a difference over a quotient using table values), 1 point for the correct value with units ('degrees Celsius per minute' attached to the numeric value).

Solution 1

(b) Mark scheme

- Left Riemann sum with the three subintervals from the table: $\int_0^{12} C(t)\,dt \approx (3-0)\cdot C(0) + (7-3)\cdot C(3) + (12-7)\cdot C(7) = 3(100) + 4(85) + 5(69) = 300 + 340 + 345 = 985$. Interpretation: $\frac{1}{12}\int_0^{12} C(t)\,dt$ is the average temperature of the coffee (in degrees Celsius) over the interval $t = 0$ to $t = 12$. 1 point for the correct form of the left Riemann sum (widths times left-endpoint values), 1 point for the numeric estimate 985, 1 point for an interpretation that names both 'average temperature' and the time interval.

Solution 1

(c)

Mark scheme

- $C(20) = C(12) + \int_{12}^{20} C'(t)\,dt = 55 - 14.670812 = 40.329188 \approx 40.329$ degrees Celsius. 1 point for setting up the definite integral of $C'(t)$ with correct limits 12 to 20, 1 point for adding the initial condition $C(12) = 55$ (symbolically or numerically), 1 point for the final numeric answer 40.329 (or 40.330) with supporting work shown.

Solution 1

(d) Mark scheme

- Because $C''(t) > 0$ on $12 < t < 20$ (from $C''(t) = \dfrac{0.2455e^{0.01t}(100 - t)}{t^2}$, numerator and denominator both positive on this interval), the rate of change of temperature $C'(t)$ is increasing on this interval — that is, the temperature of the coffee is changing at an increasing rate. The single point requires a correct answer ('increasing rate') together with a reason that references the sign of the second derivative of C (not just evaluation at one point, and not an ambiguous pronoun).

Question 4

2024 ▪ Q4

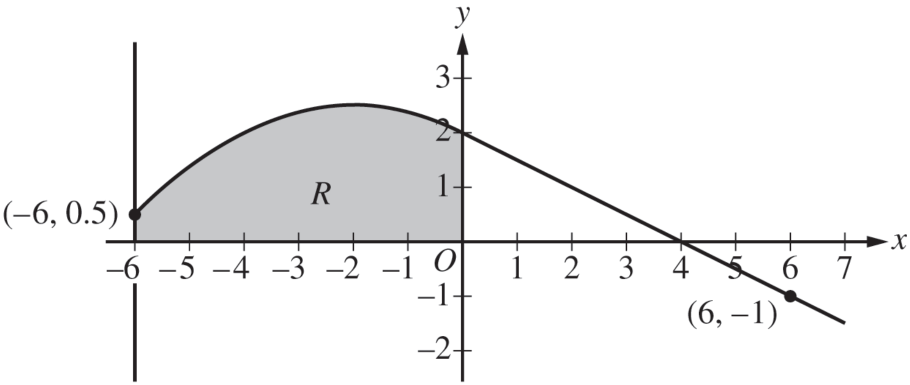
[Graph of f; x-axis labeled -6 to 7, y-axis labeled -2 to 3; the labeled point $(-6, 0.5)$ is the left endpoint of the curve; the curve rises to a horizontal tangent (peak) at $x = -2$ reaching just above $y = 2$, then descends, crossing the x-axis near $x = 4$, and continues as a straight line down to the labeled point $(6, -1)$ and beyond; the shaded region R lies in the second quadrant between the curve, the vertical line $x = -6$, and the x- and y-axes, spanning roughly $-6 \leq x \leq 0$.]
The graph of the differentiable function f, shown for $-6 \leq x \leq 7$, has a horizontal tangent at $x = -2$ and is linear for $0 \leq x \leq 7$. Let R be the region in the second quadrant bounded by the graph of f, the vertical line $x = -6$, and the x- and y-axes. Region R has area 12.

(a) The function g is defined by $g(x) = \int_0^x f(t)\,dt$. Find the values of $g(-6)$, $g(4)$, and $g(6)$.

Question 4

(b) For the function g defined in part (a), find all values of x in the interval $0 \leq x \leq 6$ at which the graph of g has a critical point. Give a reason for your answer.
(c) The function h is defined by $h(x) = \int_{-6}^{x} f'(t)\,dt$. Find the values of $h(6)$, $h'(6)$, and $h''(6)$. Show the work that leads to your answers.

Question 4



Graph of f

Solution 4

(a) Mark scheme

- Using the graph of f and $g(x) = \int_0^x f(t)\,dt$:
  $g(-6) = \int_0^{-6} f(t)\,dt = -\int_{-6}^{0} f(t)\,dt = -12$ (since region R, the area under f from -6 to 0, has area 12, given in the problem). $g(4) = \int_0^4 f(t)\,dt = \frac{1}{2}(4)(2) = 4$ (triangle area under the linear piece of f from $x = 0$, where $f(0) = 2$, down to where it crosses the x-axis at $x = 4$).
  $g(6) = \int_0^6 f(t)\,dt = \frac{1}{2}(4)(2) - \frac{1}{2}(2)(1) = 4 - 1 = 3$ (adding the negative triangle from $x = 4$ to $x = 6$, where $f(6) = -1$). 1 point each for $g(-6) = -12$, $g(4) = 4$, $g(6) = 3$ (supporting work not required, but if shown it must be correct to earn the point; unlabeled values are read in the order $g(-6), g(4), g(6)$).

Solution 4

Solution 4

(b)

Mark scheme

- By the Fundamental Theorem of Calculus, $g'(x) = f(x)$. Setting $g'(x) = f(x) = 0$ on $0 \leq x \leq 6$ gives $x = 4$ (where the graph of f crosses the x-axis, matching part (a)). Therefore the graph of g has a critical point at $x = 4$. 1 point for explicitly making the connection $g' = f$, 1 point for the answer $x = 4$ with a valid reason (e.g. $f(4) = 0$ and f changes sign there).

Solution 4

(c) Mark scheme

- $h(x) = \int_{-6}^{x} f'(t)\,dt$, so by the Fundamental Theorem of Calculus $h(6) = \int_{-6}^{6} f'(t)\,dt = f(6) - f(-6) = -1 - 0.5 = -1.5$. $h'(x) = f'(x)$, so $h'(6) = f'(6) = -\dfrac{1}{2}$ (the slope of the linear piece of f on $0 \le x \le 7$, which passes through $(0, 2)$ and $(6, -1)$). $h''(x) = f''(x)$, so $h''(6) = f''(6) = 0$ (since f is linear on $[0, 7]$, its second derivative is 0 there). 1 point for using the Fundamental Theorem of Calculus to evaluate $h(6)$ as $f(6) - f(-6)$, 1 point for the value $h(6) = -1.5$ with supporting work, 1 point for $h'(6) = -1/2$, 1 point for $h''(6) = 0$.

Question 5

2023 ▪ Q5

The functions f and g are twice differentiable. The table shown gives values of the functions and their first derivatives at selected values of x.

x	0	2	4	7
$f(x)$	10	7	4	5
$f'(x)$	$\frac{3}{2}$	-8	3	6
$g(x)$	1	2	-3	0
$g'(x)$	5	4	2	8

(a) Let h be the function defined by $h(x) = f(g(x))$. Find $h'(7)$. Show the work that leads to your answer.

(b) Let k be a differentiable function such that $k'(x) = (f(x))^2 \cdot g(x)$. Is the graph of k concave up or concave down at the point where $x = 4$? Give a reason for your answer.

(c) Let m be the function defined by $m(x) = 5x^3 + \int_0^x f'(t)\,dt$. Find $m(2)$. Show the work that leads to your answer.

Question 5

(d) Is the function m defined in part (c) increasing, decreasing, or neither at $x = 2$? Justify your answer.

Question 5

x	0	2	4	7
$f(x)$	10	7	4	5
$f'(x)$	$\frac{3}{2}$	-8	3	6
$g(x)$	1	2	-3	0
$g'(x)$	5	4	2	8

Solution 5

(a)

Mark scheme

- $h'(x) = f'(g(x)) \cdot g'(x)$ (1 pt: chain rule).
  $h'(7) = f'(g(7)) \cdot g'(7) = f'(0) \cdot 8 = \frac{3}{2} \cdot 8 = 12$ (1 pt: correct answer 12, using the table values g(7)=0, f'(0)=3/2, g'(7)=8).

Solution 5

(b)

Mark scheme

- $k''(x) = 2f(x)f'(x)g(x) + (f(x))^2 g'(x)$ (1 pt: product/chain rule applied to k'(x)=(f(x))^2 g(x)). $k''(4) = 2f(4)f'(4)g(4) + (f(4))^2 g'(4) = 2(4)(3)(-3) + (4)^2(2) = -72 + 32 = -40$ (1 pt: k''(4)=-40 with supporting work using the table values f(4)=4, f'(4)=3, g(4)=-3, g'(4)=2). Since k''(4)=-40<0 and k'' is continuous, the graph of k is concave down at x=4 (1 pt: answer with reason, consistent with the declared nonzero value of k''(4)).

Solution 5

(c)

Mark scheme

- By the Fundamental Theorem of Calculus, $m(2) = 5(2)^3 + \int_0^2 f'(t)\,dt = 40 + (f(2) - f(0)) = 40 + (7 - 10) = 37$ (1 pt: answer 37 with supporting work equivalent to $5 \cdot 8 + (f(2) - f(0))$).

Solution 5

(d)

Mark scheme

- $m'(x) = 15x^2 + f'(x)$ (1 pt: considers m'(x)).
  $m'(2) = 15(2)^2 + f'(2) = 60 + (-8) = 52$ (1 pt: m'(2) with supporting work, using f'(2)=-8 from the table; an unsupported m'(2)=52 does not earn this point). Since $m'(2) = 52 > 0$, the graph of m is increasing at x=2 (1 pt: answer with justification, consistent with the declared value of m'(2)).

Question 3

2021 ▪ Q3

[Graph showing a curve in the first quadrant starting at the origin O, rising to a rounded peak, then curving back down to meet the x-axis; x-axis and y-axis labeled, no numeric scale marks shown.]

A company designs spinning toys using the family of functions $y = cx\sqrt{4 - x^2}$, where c is a positive constant. The figure above shows the region in the first quadrant bounded by the x-axis and the graph of $y = cx\sqrt{4 - x^2}$, for some c. Each spinning toy is in the shape of the solid generated when such a region is revolved about the x-axis. Both x and y are measured in inches.

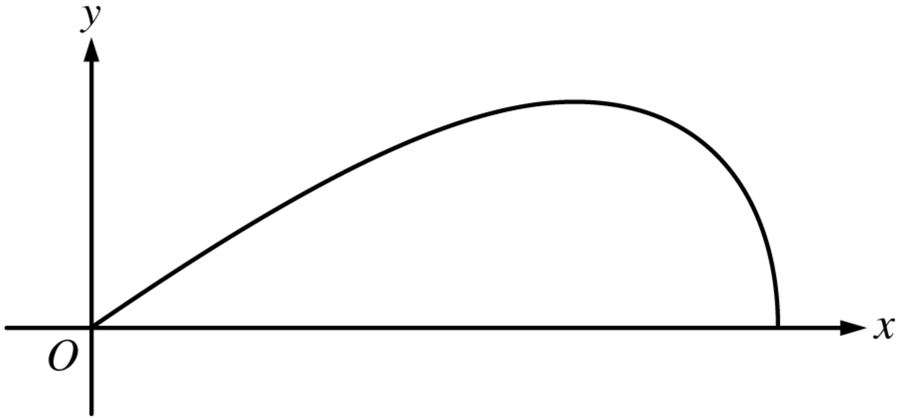
(a) Find the area of the region in the first quadrant bounded by the x-axis and the graph of $y = cx\sqrt{4 - x^2}$ for $c = 6$.

Question 3

(b) It is known that, for $y = cx\sqrt{4 - x^2}$, $\dfrac{dy}{dx} = \dfrac{c(4 - 2x^2)}{\sqrt{4 - x^2}}$. For a particular spinning toy, the radius of the largest cross-sectional circular slice is 1.2 inches. What is the value of c for this spinning toy?

(c) For another spinning toy, the volume is 2π cubic inches. What is the value of c for this spinning toy?

Question 3



Solution 3

(a) Mark scheme

- For $c = 6$: $6x\sqrt{4-x^2} = 0 \Rightarrow x = 0, 2$, so Area $= \displaystyle\int_0^2 6x\sqrt{4-x^2}\,dx$. Using $u = 4 - x^2$: $\int_0^2 6x\sqrt{4-x^2}\,dx = 3\int_0^4 u^{1/2}\,du = 2u^{3/2}\Big|_0^4 = 2\cdot 8 = 16$. The area of the region is 16 square inches. Point 1: presenting $cx\sqrt{4-x^2}$ or $6x\sqrt{4-x^2}$ as the integrand of a definite integral (limits need not be correct). Point 2: a correct antiderivative of the form $Ax\sqrt{4-x^2}$... [presented as $Au^{3/2}$-type expression] for any nonzero constant (independent of point 1). Point 3: the final answer 16 (or equivalent), only earned if point 2 was earned, using the correct limits.

Solution 3

(b) Mark scheme

- The largest cross-sectional radius occurs where $\dfrac{dy}{dx} = 0$:
  $\dfrac{dy}{dx} = \dfrac{c(4-2x^2)}{\sqrt{4-x^2}} = 0 \Rightarrow x = \sqrt{2}$. Then $y = c\sqrt{2}\sqrt{4-(\sqrt{2})^2} = 2c$. Setting $2c = 1.2$ gives $c = 0.6$. Point 1: setting $\frac{dy}{dx} = 0$ (or the equivalent $c(4-2x^2) = 0$); an unsupported $x = \sqrt{2}$ alone does not earn it. Point 2: the final answer $c = 0.6$ with supporting work (can be earned without point 1).

Solution 3

(c) Mark scheme

- Volume $= \int_0^2 \pi\left(cx\sqrt{4-x^2}\right)^2 dx = \pi c^2 \int_0^2 x^2(4-x^2)\, dx =$ $\pi c^2 \int_0^2 (4x^2 - x^4)\, dx = \pi c^2 \left(\frac{4}{3}x^3 - \frac{1}{5}x^5\right)\Big|_0^2 = \pi c^2\left(\frac{32}{3} - \frac{32}{5}\right) = \frac{64\pi c^2}{15}$. Setting $\frac{64\pi c^2}{15} = 2\pi$ gives $c^2 = \frac{15}{32}$, so $c = \sqrt{15/32}$. Point 1: integrand of the form $A(x\sqrt{4-x^2})^2$ in a definite integral, any limits, any nonzero constant A (mishandling c makes it ineligible for point 4). Point 2: correct limits $x = 0, 2$ and constant π (not 2π) as part of an integral — earnable without point 1. Point 3: correct antiderivative of the presented integrand, form $A(x\sqrt{4-x^2})^2$-derived

Solution 3

polynomial, for any nonzero A — needed for point 4. Point 4: final answer $c = \sqrt{15/32}$ (need not be simplified); cannot be earned without point 3.

Question 3

2019 ▪ Q3

The continuous function f is defined on the closed interval $-6 \leq x \leq 5$. The figure below shows a portion of the graph of f, consisting of two line segments and a quarter of a circle centered at the point $(5, 3)$. It is known that the point $\left(3,\ 3-\sqrt{5}\right)$ is on the graph of f.
[Graph of f; x-axis from -2 to 5 (gridlines shown from -2 to 5), y-axis from -1 to 3. A line segment goes from a closed point at $(-2, 1)$ down to a closed point at the origin $(0, 0)$. From $(0, 0)$ a line segment rises to a closed point at $(2, 3)$. From $(2, 3)$ a curve (quarter circle centered at $(5, 3)$) decreases and flattens out, passing through $(3, 3-\sqrt{5})$, ending at a closed point at $(5, 0)$ on the x-axis. Labeled "Graph of f".]

(a) If $\displaystyle\int_{-6}^{5} f(x)\,dx = 7$, find the value of $\displaystyle\int_{-6}^{-2} f(x)\,dx$. Show the work that leads to your answer.

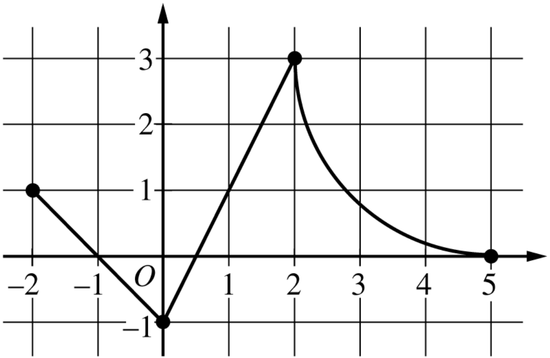
Question 3

(b) Evaluate $\int_3^5 (2f'(x) + 4)\, dx$.

(c) The function g is given by $g(x) = \int_{-2}^{x} f(t)\, dt$. Find the absolute maximum value of g on the interval $-2 \leq x \leq 5$. Justify your answer.

(d) Find $\lim_{x \to 1} \frac{10^x - 3f'(x)}{f(x) - \arctan x}$.

Question 3



Graph of f

Solution 3

(a) Mark scheme

- $\int_{-6}^{5} f(x)\,dx = \int_{-6}^{-2} f(x)\,dx + \int_{-2}^{5} f(x)\,dx$. Given $\int_{-6}^{5} f(x)\,dx = 7$ and (from the given graph of f) $\int_{-2}^{5} f(x)\,dx = 2 + \left(9 - \frac{9\pi}{4}\right) = 11 - \frac{9\pi}{4}$, so $7 = \int_{-6}^{-2} f(x)\,dx + \left(11 - \frac{9\pi}{4}\right)$, giving $\int_{-6}^{-2} f(x)\,dx = 7 - \left(11 - \frac{9\pi}{4}\right) = \frac{9\pi}{4} - 4$. Points: 1 for splitting the integral correctly ($\int_{-6}^{5} = \int_{-6}^{-2} + \int_{-2}^{5}$), 1 for the correct value of $\int_{-2}^{5} f(x)\,dx$, 1 for the final answer $\frac{9\pi}{4} - 4$.

Solution 3

(b)

Mark scheme

- $\int_3^5 (2f'(x)+4)\,dx = 2\int_3^5 f'(x)\,dx + \int_3^5 4\,dx = 2(f(5)-f(3)) + 4(5-3) = 2\left(0-(3-\sqrt{5})\right)+8 = 2(-3+\sqrt{5})+8 = 2+2\sqrt{5}$. Points: 1 for using the Fundamental Theorem of Calculus (i.e. $2(f(5)-f(3))$ term), 1 for the final answer $2+2\sqrt{5}$.

Solution 3

(c) Mark scheme

- $g(x) = \int_{-2}^{x} f(t)\,dt$, so $g'(x) = f(x)$. $g'(x) = 0$ at $x = -1$, $x = \frac{1}{2}$, $x = 5$ (the critical points/endpoints from the given graph of f on $[-2, 5]$). Evaluating g at the candidates: $g(-2) = 0$, $g(-1) = \frac{1}{2}$, $g(\frac{1}{2}) = -\frac{1}{4}$, $g(5) = 11 - \frac{9\pi}{4}$. On $-2 \leq x \leq 5$, the absolute maximum value of g is $g(5) = 11 - \frac{9\pi}{4}$. Points: 1 for $g'(x) = f(x)$, 1 for identifying $x = -1$ as a candidate (local max, since f changes from positive to negative there), 1 for the correct answer $g(5) = 11 - \frac{9\pi}{4}$ with justification (candidates test/table).

Solution 3

(d) Mark scheme

- $\lim_{x\to 1} \dfrac{10^x - 3f'(x)}{f(x) - \arctan x} = \dfrac{10^1 - 3f'(1)}{f(1) - \arctan 1} = \dfrac{10 - 3\cdot 2}{1 - \arctan 1} = \dfrac{4}{1 - \frac{\pi}{4}}$. Points: 1 for the correct final answer, $\dfrac{4}{1 - \pi/4}$.

Question 3

2018 ▪ Q3

[Graph of g: a piecewise-defined curve on axes with x from -5 to 6 (gridlines every 1 unit) and y from -3 to 8 (gridlines every 1 unit). The curve is constant at $y = -3$ for $-5 \leq x \leq -2$; rises linearly from $(-2, -3)$ to $(-1, 0)$; is constant at $y = 0$ for $-1 \leq x \leq 0$; rises linearly from $(0, 0)$ to $(1, 2)$; is constant at $y = 2$ for $1 \leq x \leq 3$; then follows a smooth curve for $3 \leq x \leq 6$ that dips down to a minimum value of 0 at $x = 4$ and rises back up, passing through approximately $(5, 1)$ and reaching $(6, 8)$.]

Graph of g

The graph of the continuous function g, the derivative of the function f, is shown above. The function g is piecewise linear for $-5 \leq x < 3$, and $g(x) = 2(x-4)^2$ for $3 \leq x \leq 6$.

(a) If $f(1) = 3$, what is the value of $f(-5)$?

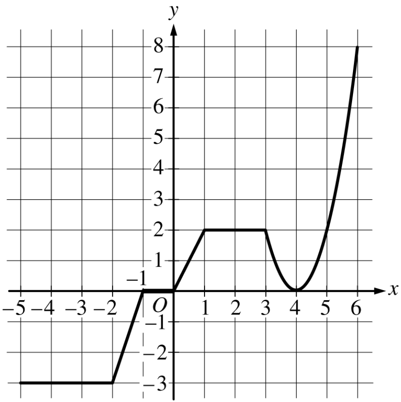
Question 3

(b) Evaluate $\int_1^6 g(x)\,dx$.

(c) For $-5 < x < 6$, on what open intervals, if any, is the graph of f both increasing and concave up? Give a reason for your answer.

(d) Find the x-coordinate of each point of inflection of the graph of f. Give a reason for your answer.

Question 3



Graph of g

Solution 3

(a)

Mark scheme

- $f(-5) = f(1) + \int_1^{-5} g(x)\,dx = f(1) - \int_{-5}^{1} g(x)\,dx = 3 - \left(-9 - \frac{3}{2} + 1\right) = 3 - \left(-\frac{19}{2}\right) = \frac{25}{2}$. (1 pt: the integral expression for $f(-5)$; 1 pt: the answer $\frac{25}{2}$.)

Solution 3

(b)

Mark scheme

- $\int_1^6 g(x)\,dx = \int_1^3 g(x)\,dx + \int_3^6 g(x)\,dx = \int_1^3 2\,dx + \int_3^6 2(x-4)^2\,dx = 4 + \left[\frac{2}{3}(x-4)^3\right]_3^6 = 4 + \frac{16}{3} - \left(-\frac{2}{3}\right) = 10$. (1 pt: splits the integral at $x = 3$; 1 pt: antiderivative $\frac{2}{3}(x-4)^3$ of $2(x-4)^2$; 1 pt: the answer 10.)

Solution 3

(c)

Mark scheme

- The graph of f is increasing and concave up on $0 < x < 1$ and $4 < x < 6$, because $f'(x) = g(x) > 0$ and $f'(x) = g(x)$ is increasing (so $f''(x) = g'(x) > 0$) on those intervals. (1 pt: the correct intervals; 1 pt: the reason.)

Solution 3

(d)

Mark scheme

- The graph of f has a point of inflection at $x = 4$ because $f'(x) = g(x)$ changes from decreasing to increasing at $x = 4$ (so f'' changes sign there). (1 pt: the answer $x = 4$; 1 pt: the reason.)

Question 3

2017 ▪ Q3

[Graph of f', labeled "Graph of f'"; consists of a semicircle and three line segments. x-axis x, y-axis y. A line segment starts at the labeled point $(-6, 2)$ and decreases linearly to the point $(-2, 0)$ on the x-axis. From $(-2, 0)$ to $(0, 0)$ the graph is a semicircle dipping below the x-axis (minimum around $(-1, -1)$) and returning to the x-axis at $(0, 0)$. From $(0, 0)$ the graph rises linearly to the labeled point $(3, 2)$. From $(3, 2)$ the graph decreases linearly to the point $(5, 0)$ on the x-axis. Gridlines/tick marks are shown at integer x-values and at $y = 1$.]

The function f is differentiable on the closed interval $[-6, 5]$ and satisfies $f(-2) = 7$. The graph of f', the derivative of f, consists of a semicircle and three line segments, as shown in the figure above.

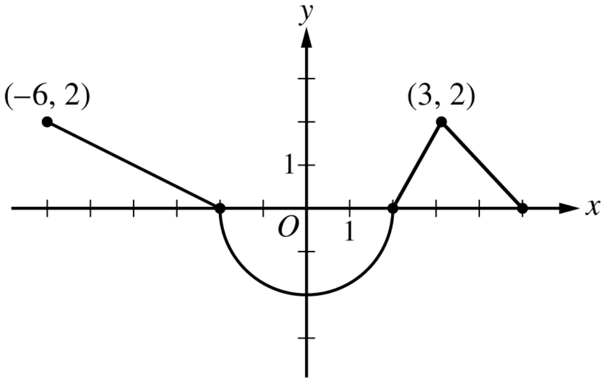
(a) Find the values of $f(-6)$ and $f(5)$.

(b) On what intervals is f increasing? Justify your answer.

Question 3

(c) Find the absolute minimum value of f on the closed interval $[-6, 5]$. Justify your answer.
(d) For each of $f''(-5)$ and $f''(3)$, find the value or explain why it does not exist.

Question 3



Graph of f'

Question 6

2016 ▪ Q6

The functions f and g have continuous second derivatives. The table above gives values of the functions and their derivatives at selected values of x.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
1	-6	3	2	8
2	2	-2	-3	0
3	8	7	6	2
6	4	5	3	-1

Question 6

(a) Let $k(x) = f(g(x))$. Write an equation for the line tangent to the graph of k at $x = 3$.

(b) Let $h(x) = \dfrac{g(x)}{f(x)}$. Find $h'(1)$.

(c) Evaluate $\displaystyle\int_1^3 f'(2x)\,dx$.

Question 6

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
1	–6	3	2	8
2	2	–2	–3	0
3	8	7	6	2
6	4	5	3	–1

Solution 6

(a)

Mark scheme

- k(x)=f(g(x)). k(3)=f(g(3))=f(6)=4. By the chain rule, k'(x)=f'(g(x))·$g'(x), sok'(3) = f'(g(3)) \cdot g'(3) = f'(6) \cdot g'(3) = 5 \cdot 2 = 10. Tangentline: y = 10(x-3) + 4. Points: 2forcorrectlyfindingtheslopek'(3) = 10(chainruleusingtablevalues), 1forthecorrecttangentlineequation.$

Solution 6

(b) Mark scheme

- h(x)=g(x)/f(x). By the quotient rule: h'(x) $= \dfrac{f(x)g'(x) - g(x)f'(x)}{[f(x)]^2}. h'(1) = \dfrac{f(1)g'(1) - g(1)f'(1)}{[f(1)]^2} = \dfrac{(-6)(8) - (2)(3)}{(-6)^2} = \dfrac{-48-6}{36} = \dfrac{-54}{36} = -\dfrac{3}{2}. Points: 2 forthecorrectexpressionforh'(1) usingthequotientruleandtablevalues, 1 forthecorrectfinalanswer($

Solution 6

(c) Mark scheme

- $\int_1^3 f'(2x)\,dx = \frac{1}{2}\left[f(2x)\right]_1^3 = \frac{1}{2}\left[f(6) - f(2)\right] = \frac{1}{2}\left[5 - (-2)\right] = \frac{1}{2}(7) = \frac{7}{2}. Points: 2 for the correct antiderivative via u-substitution (\tfrac{1}{2}f(2x)), 1 for the correct final answer (7/2).$

Question 5

2015 ▪ Q5

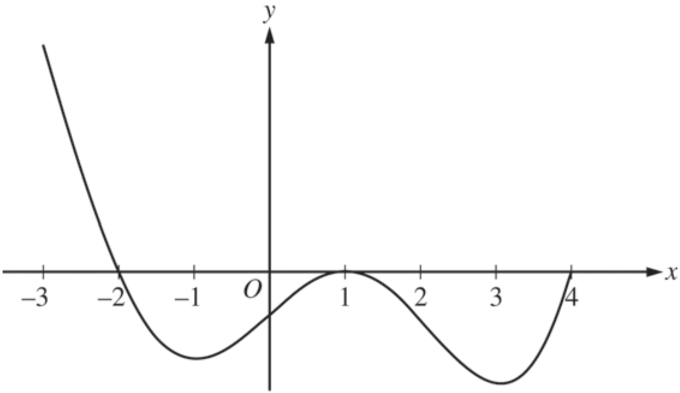
[Graph of f': axes labeled y (vertical) and x (horizontal), with tick marks on the x-axis at $-3, -2, -1, O, 1, 2, 3, 4$. The curve begins high on the left (above $x = -3$), decreases steeply, crosses the x-axis just before $x = -1$, continues down to a local minimum between $x = -1$ and O, rises back up crossing the x-axis near $x = 1$, reaches a local maximum between $x = 1$ and $x = 2$, then decreases again crossing the x-axis before $x = 3$, reaches a local minimum between $x = 3$ and $x = 4$, then rises steeply up to $x = 4$. Labeled "Graph of f'".]

The figure above shows the graph of f', the derivative of a twice-differentiable function f, on the interval $[-3, 4]$. The graph of f' has horizontal tangents at $x = -1$, $x = 1$, and $x = 3$. The areas of the regions bounded by the x-axis and the graph of f' on the intervals $[-2, 1]$ and $[1, 4]$ are 9 and 12, respectively.

Question 5

(a) Find all x-coordinates at which f has a relative maximum. Give a reason for your answer.
(b) On what open intervals contained in $-3 < x < 4$ is the graph of f both concave down and decreasing? Give a reason for your answer.
(c) Find the x-coordinates of all points of inflection for the graph of f. Give a reason for your answer.
(d) Given that $f(1) = 3$, write an expression for $f(x)$ that involves an integral. Find $f(4)$ and $f(-2)$.

Question 5



Graph of f'

Solution 5

(a)

Mark scheme

- $f'(x) = 0$ at $x = -2$, $x = 1$, and $x = 4$. $f'(x)$ changes from positive to negative only at $x = -2$. Therefore f has a relative maximum at $x = -2$ (this is the only x-coordinate at which f has a relative maximum). Points: 1 for identifying $x = -2$, 1 for the answer with correct reason (sign change of f' from $+$ to $-$).

Solution 5

(b)

Mark scheme

- The graph of f is concave down and decreasing on the intervals $-2 < x < -1$ and $1 < x < 3$, because f' is decreasing and negative on these intervals. Points: 1 for identifying the correct intervals $(-2, -1)$ and $(1, 3)$, 1 for the correct reason (f' decreasing gives concave down; f' negative gives decreasing).

Solution 5

(c)

Mark scheme

- The graph of f has points of inflection at $x = -1$ and $x = 3$, because f' changes from decreasing to increasing at these points. The graph of f also has a point of inflection at $x = 1$, because f' changes from increasing to decreasing at this point. So the inflection x-coordinates are -1, 1, and 3. Points: 1 for identifying $x = -1$, 1, and 3, 1 for the correct reason (f' changes from increasing to decreasing, or decreasing to increasing, at each point).

Solution 5

(d)

Mark scheme

- f(x) = 3 + $\int$ ^x f'(t) dt (using f(1) = 3). f(4) = 3 + $\int$ ^4 f'(t) dt = 3 + (−12) = −9 (the integral is the negative of the area 12 on [1, 4], since f' is negative there). f(−2) = 3 + $\int$ ^(−2) f'(t) dt = 3 − $\int$_(−2)^1 f'(t) dt = 3 − (−9) = 12 (the integral over [−2, 1] is the negative of the area 9, since f' is negative there). Points: 1 for the correct integrand f'(t) in an expression for f, 1 for a correct expression for f(x) (f(x) = 3 + $\int$ ^x f'(t) dt), 1 for both correct final values f(4) = −9 and f(−2) = 12.