Past-paper walk-through

Applying Properties of Definite Integrals ▪ 6.6

eXercise

Questions in this set

- 2019 Q3
- 2018 Q3

Reading the mark scheme

- **B1** a mark that stands on its own – the point is either made or not.
- **M1** a *method* mark: the right approach, even if the number is wrong.
- **A1** an *answer* mark, and it usually needs its M mark first.
- **C1** a working mark on the way to the answer.
- // separates answers the examiner accepts equally.
- **ecf** = error carried forward: a later part is marked on your own earlier answer.
- **owtte** = “or words to that effect” – the wording need not match.

Question 3

2019 ▪ Q3

The continuous function *f* is defined on the closed interval $-6 \leq x \leq 5$. The figure below shows a portion of the graph of *f*, consisting of two line segments and a quarter of a circle centered at the point $(5, 3)$. It is known that the point $(3,\ 3-\sqrt{5})$ is on the graph of *f*.
[Graph of *f*; *x*-axis from -2 to 5 (gridlines shown from -2 to 5), *y*-axis from -1 to 3. A line segment goes from a closed point at $(-2, 1)$ down to a closed point at the origin $(0, 0)$. From $(0, 0)$ a line segment rises to a closed point at $(2, 3)$. From $(2, 3)$ a curve (quarter circle centered at $(5, 3)$) decreases and flattens out, passing through $(3, 3-\sqrt{5})$, ending at a closed point at $(5, 0)$ on the *x*-axis. Labeled "Graph of *f*".]
(a) If $\int_{-6}^{5} f(x)\,dx = 7$, find the value of $\int_{-6}^{-2} f(x)\,dx$. Show the work that leads to your answer.

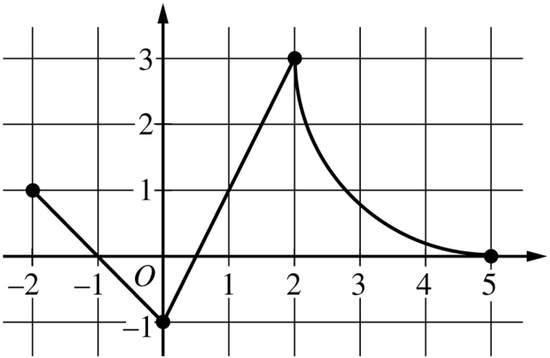
Question 3

(b) Evaluate $\int_3^5 (2f'(x) + 4)\, dx$.

(c) The function g is given by $g(x) = \int_{-2}^{x} f(t)\, dt$. Find the absolute maximum value of g on the interval $-2 \le x \le 5$. Justify your answer.

(d) Find $\lim_{x \to 1} \dfrac{10^x - 3f'(x)}{f(x) - \arctan x}$.

Question 3



Graph of f

Solution 3

(a) Mark scheme

- $\int_{-6}^{5} f(x)\,dx = \int_{-6}^{-2} f(x)\,dx + \int_{-2}^{5} f(x)\,dx$. Given $\int_{-6}^{5} f(x)\,dx = 7$ and (from the given graph of f) $\int_{-2}^{5} f(x)\,dx = 2 + \left(9 - \frac{9\pi}{4}\right) = 11 - \frac{9\pi}{4}$, so $7 = \int_{-6}^{-2} f(x)\,dx + \left(11 - \frac{9\pi}{4}\right)$, giving $\int_{-6}^{-2} f(x)\,dx = 7 - \left(11 - \frac{9\pi}{4}\right) = \frac{9\pi}{4} - 4$. Points: 1 for splitting the integral correctly ($\int_{-6}^{5} = \int_{-6}^{-2} + \int_{-2}^{5}$), 1 for the correct value of $\int_{-2}^{5} f(x)\,dx$, 1 for the final answer $\frac{9\pi}{4} - 4$.

Solution 3

(b)

Mark scheme

- $\int_3^5 (2f'(x) + 4)\,dx = 2\int_3^5 f'(x)\,dx + \int_3^5 4\,dx = 2(f(5) - f(3)) + 4(5-3) = 2\left(0 - (3 - \sqrt{5})\right) + 8 = 2(-3 + \sqrt{5}) + 8 = 2 + 2\sqrt{5}$. Points: 1 for using the Fundamental Theorem of Calculus (i.e. $2(f(5) - f(3))$ term), 1 for the final answer $2 + 2\sqrt{5}$.

Solution 3

(c) Mark scheme

- $g(x) = \int_{-2}^{x} f(t)\,dt$, so $g'(x) = f(x)$. $g'(x) = 0$ at $x = -1,\ x = \frac{1}{2},\ x = 5$ (the critical points/endpoints from the given graph of f on $[-2, 5]$). Evaluating g at the candidates: $g(-2) = 0$, $g(-1) = \frac{1}{2}$, $g(\frac{1}{2}) = -\frac{1}{4}$, $g(5) = 11 - \frac{9\pi}{4}$. On $-2 \leq x \leq 5$, the absolute maximum value of g is $g(5) = 11 - \frac{9\pi}{4}$. Points: 1 for $g'(x) = f(x)$, 1 for identifying $x = -1$ as a candidate (local max, since f changes from positive to negative there), 1 for the correct answer $g(5) = 11 - \frac{9\pi}{4}$ with justification (candidates test/table).

Solution 3

(d) Mark scheme

- $\lim_{x\to 1} \dfrac{10^x - 3f'(x)}{f(x) - \arctan x} = \dfrac{10^1 - 3f'(1)}{f(1) - \arctan 1} = \dfrac{10 - 3\cdot 2}{1 - \arctan 1} = \dfrac{4}{1-\frac{\pi}{4}}$. Points: 1 for the correct final answer, $\dfrac{4}{1-\pi/4}$.

Question 3

2018 ▪ Q3

[Graph of g: a piecewise-defined curve on axes with x from -5 to 6 (gridlines every 1 unit) and y from -3 to 8 (gridlines every 1 unit). The curve is constant at $y = -3$ for $-5 \leq x \leq -2$; rises linearly from $(-2, -3)$ to $(-1, 0)$; is constant at $y = 0$ for $-1 \leq x \leq 0$; rises linearly from $(0, 0)$ to $(1, 2)$; is constant at $y = 2$ for $1 \leq x \leq 3$; then follows a smooth curve for $3 \leq x \leq 6$ that dips down to a minimum value of 0 at $x = 4$ and rises back up, passing through approximately $(5, 1)$ and reaching $(6, 8)$.]

Graph of g

The graph of the continuous function g, the derivative of the function f, is shown above. The function g is piecewise linear for $-5 \leq x < 3$, and $g(x) = 2(x-4)^2$ for $3 \leq x \leq 6$.

(a) If $f(1) = 3$, what is the value of $f(-5)$?

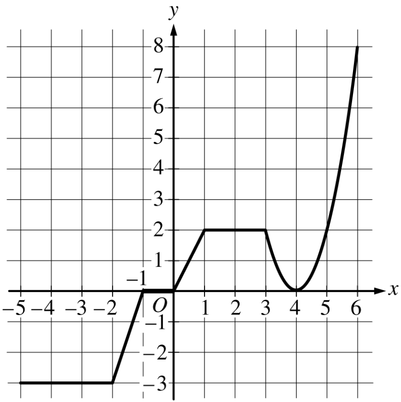
Question 3

(b) Evaluate $\int_1^6 g(x)\,dx$.

(c) For $-5 < x < 6$, on what open intervals, if any, is the graph of f both increasing and concave up? Give a reason for your answer.

(d) Find the x-coordinate of each point of inflection of the graph of f. Give a reason for your answer.

Question 3



Graph of g

Solution 3

(a)

Mark scheme

- $f(-5) = f(1) + \int_1^{-5} g(x)\,dx = f(1) - \int_{-5}^{1} g(x)\,dx = 3 - \left(-9 - \frac{3}{2} + 1\right) = 3 - \left(-\frac{19}{2}\right) = \frac{25}{2}$. (1 pt: the integral expression for $f(-5)$; 1 pt: the answer $\frac{25}{2}$.)

Solution 3

(b)

Mark scheme

- $\int_1^6 g(x)\,dx = \int_1^3 g(x)\,dx + \int_3^6 g(x)\,dx = \int_1^3 2\,dx + \int_3^6 2(x-4)^2\,dx = 4 + \left[\frac{2}{3}(x-4)^3\right]_3^6 = 4 + \frac{16}{3} - \left(-\frac{2}{3}\right) = 10$. (1 pt: splits the integral at $x = 3$; 1 pt: antiderivative $\frac{2}{3}(x-4)^3$ of $2(x-4)^2$; 1 pt: the answer 10.)

Solution 3

(c)

Mark scheme

- The graph of f is increasing and concave up on $0 < x < 1$ and $4 < x < 6$, because $f'(x) = g(x) > 0$ and $f'(x) = g(x)$ is increasing (so $f''(x) = g'(x) > 0$) on those intervals. (1 pt: the correct intervals; 1 pt: the reason.)

Solution 3

(d)

Mark scheme

- The graph of f has a point of inflection at $x = 4$ because $f'(x) = g(x)$ changes from decreasing to increasing at $x = 4$ (so f'' changes sign there). (1 pt: the answer $x = 4$; 1 pt: the reason.)