Past-paper walk-through

The Fundamental Theorem of Calculus and Accumulation Functions ▪ 6.4

eXercise

Questions in this set

- 2026 Q6
- 2025 Q1
- 2025 Q4
- 2024 Q4
- 2022 Q1
- 2022 Q3
- 2021 Q4
- 2019 Q3
- 2015 Q1

Reading the mark scheme

- **B1** a mark that stands on its own – the point is either made or not.
- **M1** a *method* mark: the right approach, even if the number is wrong.
- **A1** an *answer* mark, and it usually needs its M mark first.
- **C1** a working mark on the way to the answer.
- // separates answers the examiner accepts equally.
- **ecf** = error carried forward: a later part is marked on your own earlier answer.
- **owtte** = "or words to that effect" – the wording need not match.

Question 6

2026 ▪ Q6

The function f is twice differentiable. The table gives values of f and its derivative f' at selected values of x.

x	**0**	**2**	**3**	**6**
$f(x)$	-1	3	8	5
$f'(x)$	-5	4	9	-2

Question 6

(a) Find $\lim_{x\to 2} \frac{f(x)}{x}$, or state that the limit does not exist.

(b) Let $g(x) = f(f(x))$. Find $g'(2)$. Show the work that leads to your answer.

(c) Let h be a differentiable function such that $h(0) = 10$ and $h'(x) = f'(3x)$. Find $h(2)$. Show the work that leads to your answer.

(d)(i) Let k be the function defined by $k(x) = \int_0^x t^2 f(t)\,dt$. Find $k'(x)$.

(d)(ii) Find $k''(3)$. Show the work that leads to your answer.

Question 6

x	0	2	3	6
$f(x)$	-1	3	8	5
$f'(x)$	-5	4	9	-2

Question 1

2025 ▪ Q1

An invasive species of plant appears in a fruit grove at time $t = 0$ and begins to spread. The function C defined by $C(t) = 7.6\arctan(0.2t)$ models the number of acres in the fruit grove affected by the species t weeks after the species appears. It can be shown that $C'(t) = \dfrac{38}{25 + t^2}$.

(Note: Your calculator should be in radian mode.)

(a) Find the average number of acres affected by the invasive species from time $t = 0$ to time $t = 4$ weeks. Show the setup for your calculations.

Question 1

2025 ▪ Q1

An invasive species of plant appears in a fruit grove at time $t = 0$ and begins to spread. The function C defined by $C(t) = 7.6\arctan(0.2t)$ models the number of acres in the fruit grove affected by the species t weeks after the species appears. It can be shown that $C'(t) = \frac{38}{25 + t^2}$.

(Note: Your calculator should be in radian mode.)

(b) Find the time t when the instantaneous rate of change of C equals the average rate of change of C over the time interval $0 \le t \le 4$. Show the setup for your calculations.

Question 1

2025 ▪ Q1

An invasive species of plant appears in a fruit grove at time $t = 0$ and begins to spread. The function C defined by $C(t) = 7.6\arctan(0.2t)$ models the number of acres in the fruit grove affected by the species t weeks after the species appears. It can be shown that $C'(t) = \dfrac{38}{25 + t^2}$. (Note: Your calculator should be in radian mode.)

(c) Assume that the invasive species continues to spread according to the given model for all times $t > 0$. Write a limit expression that describes the end behavior of the rate of change in the number of acres affected by the species. Evaluate this limit expression.

Question 1

2025 ▪ Q1

An invasive species of plant appears in a fruit grove at time $t = 0$ and begins to spread. The function C defined by $C(t) = 7.6\arctan(0.2t)$ models the number of acres in the fruit grove affected by the species t weeks after the species appears. It can be shown that $C'(t) = \dfrac{38}{25 + t^2}$.

(Note: Your calculator should be in radian mode.)

(d) At time $t = 4$ weeks after the invasive species appears in the fruit grove, measures are taken to counter the spread of the species. The function A, defined by $A(t) = C(t) - \int_4^t 0.1 \cdot \ln(x)\, dx$, models the number of acres affected by the species over the time interval $4 \le t \le 36$. At what time t, for $4 \le t \le 36$, does A attain its maximum value? Justify your answer.

Solution 1

(a) Mark scheme

- Average value $= \dfrac{1}{4-0}\int_0^4 C(t)\,dt = \dfrac{1}{4}(11.112896) = 2.778224$, so the average number of acres affected from $t=0$ to $t=4$ is about 2.778 acres. Point 1: correct integral $\dfrac{1}{4-0}\int_0^4 C(t)\,dt$, with or without dt, with evidence of division by 4. Point 2: correct answer, accurate to 3 decimal places (2.778 or 2.778224), with or without supporting work.

Solution 1

(b) Mark scheme

- Average rate of change of C over $[0,4]$: $\dfrac{C(4)-C(0)}{4-0} = 1.282008$. Set $C'(t) = \dfrac{38}{25+t^2}$ equal to this: $\dfrac{38}{25+t^2} = 1.282008 \Rightarrow t = 2.154298 \approx 2.154$ weeks. Point 3: presents the average rate of change expression or value (e.g. $\dfrac{C(4)-C(0)}{4-0}$, $\dfrac{C(4)}{4}$, $\dfrac{5.128}{4}$, or 1.282). Point 4: correct answer $t \approx 2.154$, supported by the equation $C'(t) =$ average rate of change.

Solution 1

(c)

Mark scheme

- End behavior: $\lim_{t\to\infty} C'(t) = \lim_{t\to\infty} \frac{38}{25+t^2} = 0$. Point 5: correct limit expression, either $\lim_{t\to\infty} C'(t)$ or $\lim_{t\to\infty} C(t)$. Point 6: correct value 0 (only earnable if the limit expression used C', not C).

Solution 1

(d)

Mark scheme

- $A'(t) = C'(t) - 0.1 \ln t$. Set $A'(t) = 0$: $C'(t) = 0.1 \ln t \Rightarrow t = 11.441700$. Candidates test: $A(4) = 5.128031$, $A(11.4417) = 7.316978$, $A(36) = 1.743056$. The maximum is at $t = 11.442$ (or 11.441) weeks. Point 7: considers $A'(t) = 0$ (i.e., $C'(t) - 0.1 \ln t = 0$). Point 8: justification via a correct global (candidates-test) argument evaluating A at $t = 4$, $t = 11.4417$, and $t = 36$ (or an equivalent sign-analysis argument that $A'(t) > 0$ then $A'(t) < 0$). Point 9: correct answer $t \approx 11.442$ weeks, supported by work.

Question 4

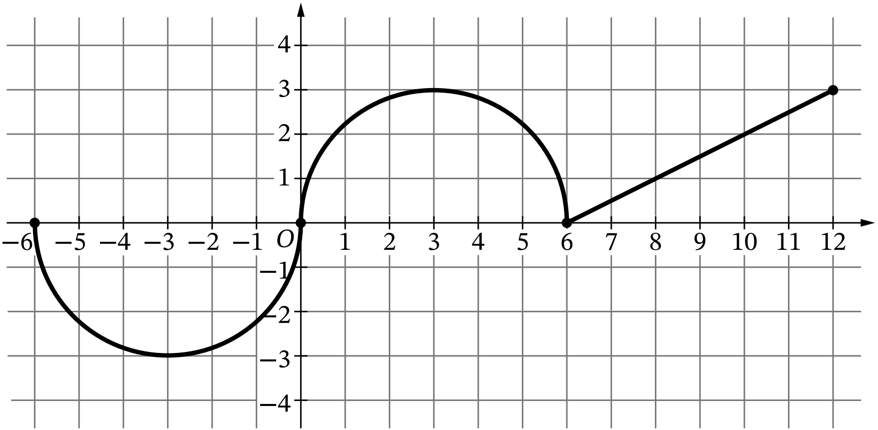
2025 ▪ Q4

The continuous function f is defined on the closed interval $-6 \leq x \leq 12$. The graph of f, consisting of two semicircles and one line segment, is shown in the figure.
[Graph of f: On the interval $-6 \leq x \leq 0$, the graph is a semicircle below the x-axis with endpoints at $(-6, 0)$ and $(0, 0)$, reaching a minimum near $(-3, -3)$. On the interval $0 \leq x \leq 6$, the graph is a semicircle above the x-axis with endpoints at $(0, 0)$ and $(6, 0)$, reaching a maximum near $(3, 3)$. On the interval $6 \leq x \leq 12$, the graph is a line segment from $(6, 0)$ to $(12, 3)$. The horizontal axis is labeled from -6 to 12 and the vertical axis from -4 to 4.]
Graph of f
Let g be the function defined by $g(x) = \int_6^x f(t)\, dt$.
(a) Find $g'(8)$. Give a reason for your answer.

Question 4

(b) Find all values of x in the open interval $-6 < x < 12$ at which the graph of g has a point of inflection. Give a reason for your answer.
(c) Find $g(12)$ and $g(0)$. Label your answers.
(d) Find the value of x at which g attains an absolute minimum on the closed interval $-6 \leq x \leq 12$. Justify your answer.

Question 4



Graph of f

Solution 4

(a)

Mark scheme

- By the Fundamental Theorem of Calculus, $g'(x) = f(x)$, so $g'(8) = f(8) = 1$ (read from the graph of f at $x = 8$, which is on the line segment). Point 1: considers $g'(x) = f(x)$ (or $g' = f$, or $g'(8) = f(8)$). Point 2: correct answer $g'(8) = 1$ (an implied application of the Fundamental Theorem of Calculus, e.g. just "$g'(8) = 1$" or "$f(8) = 1$", is also sufficient for this point alone).

Solution 4

(b) Mark scheme

- The graph of g has a point of inflection where $g'' = f'$ changes sign, i.e. where f changes from decreasing to increasing or vice versa (equivalently, where f attains a relative extremum, or where the slope of f changes sign). From the graph, this happens at $x = -3$, $x = 3$, and $x = 6$. Point 3: correct answer of exactly $x = -3$, $x = 3$, and $x = 6$ (if any other/additional values are declared, neither this point nor the reason point is earned; consideration of $x = -6$ or $x = 12$ does not impact scoring). Point 4: reason correctly tied to the given graph of f (e.g. "f changes from increasing to decreasing or decreasing to increasing there," or "f attains relative extrema there"; a reason relying on an ambiguous term like "the function" or "the graph" does not earn this point).

Solution 4

(c) Mark scheme

- $g(12) = \int_6^{12} f(t)\,dt = \frac{1}{2}(6)(3) = 9$ (area of the triangular line-segment region). $g(0) = \int_6^0 f(x)\,dx = -\int_0^6 f(x)\,dx = -\frac{\pi}{2}(3)^2 = -\frac{9\pi}{2}$ (negative of the semicircle area, since integrating from 6 down to 0). Point 5: $g(12) = 9$, with or without supporting work, correctly labeled as $g(12)$ (an unlabeled value does not earn the point). Point 6: $g(0) = -\frac{9\pi}{2}$, with or without supporting work, correctly labeled as $g(0)$.

Solution 4

(d) Mark scheme

- g attains a minimum on $[-6, 12]$ either where $g'(x) = f(x) = 0$ or at an endpoint: candidate points are $x = -6, 0, 6, 12$. Candidates test: $g(-6) = 0$, $g(0) = -\dfrac{9\pi}{2} \approx -14.137$, $g(6) = 0$, $g(12) = 9$. The absolute minimum value is attained at $x = 0$. Point 7: considers $g'(x) = 0$ or $f(x) = 0$ (not earned by just presenting $x = 0$ and $x = 6$ with no justification). Point 8: justification via a correct global (candidates-test) argument evaluating g at all four candidate points $-6, 0, 6, 12$ (values for $g(0)$ and $g(12)$ may be imported from part C); alternatively, an argument that $g'(x) \leq 0$ (or $f(x) \leq 0$) for $-6 \leq x < 0$ and $g'(x) \geq 0$ (or $f(x) \geq 0$) for $0 < x \leq 12$. Point 9: correct answer $x = 0$; requires

Solution 4

Point 8 (a local-argument response, e.g. First/Second Derivative Test, or an incorrect global argument, does not earn Point 8 but remains eligible for Point 9 with the correct answer).

Question 4

2024 ▪ Q4

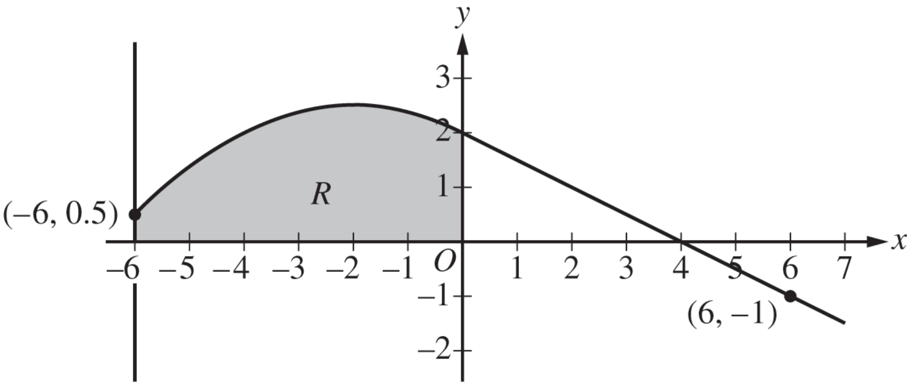
[Graph of f; x-axis labeled -6 to 7, y-axis labeled -2 to 3; the labeled point $(-6, 0.5)$ is the left endpoint of the curve; the curve rises to a horizontal tangent (peak) at $x = -2$ reaching just above $y = 2$, then descends, crossing the x-axis near $x = 4$, and continues as a straight line down to the labeled point $(6, -1)$ and beyond; the shaded region R lies in the second quadrant between the curve, the vertical line $x = -6$, and the x- and y-axes, spanning roughly $-6 \leq x \leq 0$.]
The graph of the differentiable function f, shown for $-6 \leq x \leq 7$, has a horizontal tangent at $x = -2$ and is linear for $0 \leq x \leq 7$. Let R be the region in the second quadrant bounded by the graph of f, the vertical line $x = -6$, and the x- and y-axes. Region R has area 12.
(a) The function g is defined by $g(x) = \int_0^x f(t)\, dt$. Find the values of $g(-6)$, $g(4)$, and $g(6)$.

Question 4

(b) For the function g defined in part (a), find all values of x in the interval $0 \leq x \leq 6$ at which the graph of g has a critical point. Give a reason for your answer.

(c) The function h is defined by $h(x) = \int_{-6}^{x} f'(t)\,dt$. Find the values of $h(6)$, $h'(6)$, and $h''(6)$. Show the work that leads to your answers.

Question 4



Graph of f

Solution 4

(a) Mark scheme

- Using the graph of f and $g(x) = \int_0^x f(t)\,dt$:
  $g(-6) = \int_0^{-6} f(t)\,dt = -\int_{-6}^0 f(t)\,dt = -12$ (since region R, the area under f from -6 to 0, has area 12, given in the problem). $g(4) = \int_0^4 f(t)\,dt = \frac{1}{2}(4)(2) = 4$ (triangle area under the linear piece of f from $x = 0$, where $f(0) = 2$, down to where it crosses the x-axis at $x = 4$).
  $g(6) = \int_0^6 f(t)\,dt = \frac{1}{2}(4)(2) - \frac{1}{2}(2)(1) = 4 - 1 = 3$ (adding the negative triangle from $x = 4$ to $x = 6$, where $f(6) = -1$). 1 point each for $g(-6) = -12$, $g(4) = 4$, $g(6) = 3$ (supporting work not required, but if shown it must be correct to earn the point; unlabeled values are read in the order $g(-6), g(4), g(6)$).

Solution 4

Solution 4

(b)

Mark scheme

- By the Fundamental Theorem of Calculus, $g'(x) = f(x)$. Setting $g'(x) = f(x) = 0$ on $0 \leq x \leq 6$ gives $x = 4$ (where the graph of f crosses the x-axis, matching part (a)). Therefore the graph of g has a critical point at $x = 4$. 1 point for explicitly making the connection $g' = f$, 1 point for the answer $x = 4$ with a valid reason (e.g. $f(4) = 0$ and f changes sign there).

Solution 4

(c) Mark scheme

- $h(x) = \int_{-6}^{x} f'(t)\,dt$, so by the Fundamental Theorem of Calculus $h(6) = \int_{-6}^{6} f'(t)\,dt = f(6) - f(-6) = -1 - 0.5 = -1.5$. $h'(x) = f'(x)$, so $h'(6) = f'(6) = -\dfrac{1}{2}$ (the slope of the linear piece of f on $0 \leq x \leq 7$, which passes through $(0, 2)$ and $(6, -1)$). $h''(x) = f''(x)$, so $h''(6) = f''(6) = 0$ (since f is linear on $[0, 7]$, its second derivative is 0 there). 1 point for using the Fundamental Theorem of Calculus to evaluate $h(6)$ as $f(6) - f(-6)$, 1 point for the value $h(6) = -1.5$ with supporting work, 1 point for $h'(6) = -1/2$, 1 point for $h''(6) = 0$.

Question 1

2022 ▪ Q1

From 5 A.M. to 10 A.M., the rate at which vehicles arrive at a certain toll plaza is given by $A(t) = 450\sqrt{\sin(0.62t)}$, where t is the number of hours after 5 A.M. and $A(t)$ is measured in vehicles per hour. Traffic is flowing smoothly at 5 A.M. with no vehicles waiting in line.

(a) Write, but do not evaluate, an integral expression that gives the total number of vehicles that arrive at the toll plaza from 6 A.M. ($t = 1$) to 10 A.M. ($t = 5$).

(b) Find the average value of the rate, in vehicles per hour, at which vehicles arrive at the toll plaza from 6 A.M. ($t = 1$) to 10 A.M. ($t = 5$).

(c) Is the rate at which vehicles arrive at the toll plaza at 6 A.M. ($t = 1$) increasing or decreasing? Give a reason for your answer.

(d) A line forms whenever $A(t) \geq 400$. The number of vehicles in line at time t, for $a \leq t \leq 4$, is given by

Question 1

$$N(t) = \int_a^t (A(x) - 400)\, dx,$$

where a is the time when a line first begins to form. To the nearest whole number, find the greatest number of vehicles in line at the toll plaza in the time interval $a \leq t \leq 4$. Justify your answer.

Solution 1

(a)

Mark scheme

- The total number of vehicles arriving from 6 A.M. to 10 A.M. is given by the definite integral $\int_1^5 A(t)\,dt$. Full credit requires a definite integral with the correct lower limit $t = 1$ and upper limit $t = 5$; the integrand should be $A(t)$ (equivalently $|A(t)|$, since $A(t) \geq 0$ on this interval). Do not evaluate the integral.

Solution 1

(b) Mark scheme

- Average value $= \dfrac{1}{5-1}\int_1^5 A(t)\,dt = \dfrac{1}{4}(1502.147865) = 375.536966.$ The average rate at which vehicles arrive at the toll plaza from 6 A.M. to 10 A.M. is about 375.537 (or 375.536) vehicles per hour. One point for using the correct average value formula $\frac{1}{b-a}\int_a^b A(t)\,dt$ with $a = 1,\ b = 5$; one point for the correct answer, accurate to three decimal places.

Solution 1

(c)

Mark scheme

- $A'(1) = 148.947272 > 0$, so the rate at which vehicles arrive at the toll plaza is increasing at $t = 1$ (6 A.M.). One point for considering $A'(1)$; one point for the correct answer (increasing) together with the reason $A'(1) > 0$.

Solution 1

(d)

Mark scheme

- $N'(t) = A(t) - 400 = 0 \Rightarrow A(t) = 400 \Rightarrow t = a = 1.469372$ or $t = b = 3.597713$. Evaluating $N(t) = \int_a^t (A(x) - 400)\,dx$ at the critical points and the right endpoint: $N(a) = 0$, $N(b) = 71.254129$, $N(4) = 62.338346$. The greatest number of vehicles in line is 71 (attained at $t = b$). Points: (1) considers $N'(t) = 0$; (2) correctly finds both $t = a$ and $t = b$; (3) correct answer of 71 (or 71.254...); (4) justification via a candidates-test table comparing $N(a)$, $N(b)$, and $N(4)$.

Question 3

2022 ▪ Q3

[Graph titled "Graph of f'": a coordinate plane with x-axis from 0 to 7 (integer gridlines at $1, 2, 3, 4, 5, 6, 7$) and y-axis from -2 to 2 (gridlines at $-2, -1, 1, 2$). The graph consists of a semicircle below the x-axis from $x = 0$ to $x = 4$, dipping down to a minimum of about -2 near $x = 2$, returning to 0 at $x = 4$. From $x = 4$ to $x = 6$ a line segment rises from $(4, 0)$ to the marked point $(6, 2)$. From $x = 6$ to $x = 7$ a line segment falls from $(6, 2)$ to the marked point $(7, 1)$.]

Let f be a differentiable function with $f(4) = 3$. On the interval $0 \leq x \leq 7$, the graph of f', the derivative of f, consists of a semicircle and two line segments, as shown in the figure above.

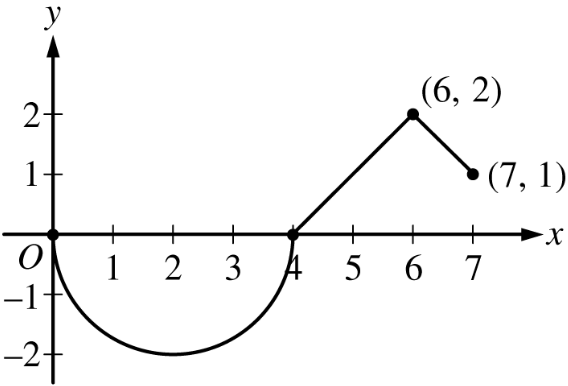
(a) Find $f(0)$ and $f(5)$.

(b) Find the x-coordinates of all points of inflection of the graph of f for $0 < x < 7$. Justify your answer.

Question 3

(c) Let g be the function defined by $g(x) = f(x) - x$. On what intervals, if any, is g decreasing for $0 \leq x \leq 7$? Show the analysis that leads to your answer.
(d) For the function g defined in part (c), find the absolute minimum value on the interval $0 \leq x \leq 7$. Justify your answer.

Question 3



Graph of f'

Solution 3

(a)

Mark scheme

- $f(0) = f(4) + \int_4^0 f'(x)\,dx = 3 - \int_0^4 f'(x)\,dx = 3 + 2\pi$ (the semicircle over $[0,4]$ has radius 2 and lies below the axis, so $\int_0^4 f'(x)\,dx = -2\pi$). $f(5) = f(4) + \int_4^5 f'(x)\,dx = 3 + \frac{1}{2} = \frac{7}{2}$. Points: (1) area of either region ($\int_0^4 f'$ or $\int_4^5 f'$); (2) correct $f(0) = 3 + 2\pi$; (3) correct $f(5) = 7/2$.

Solution 3

(b)

Mark scheme

- The graph of f has points of inflection at $x = 2$ and $x = 6$, because f' changes from decreasing to increasing at $x = 2$ (the bottom of the semicircle) and from increasing to decreasing at $x = 6$ (the corner/peak of the graph of f' at $(6, 2)$). Points: (1) states both $x = 2$ and $x = 6$; (2) correct justification tied to the graph of f' (e.g., discussing where the slope/behavior of f' changes, or citing $x = 2, 6$ as locations of local extrema on the graph of f').

Solution 3

(c)

Mark scheme

- $g(x) = f(x) - x \Rightarrow g'(x) = f'(x) - 1$. Since $g'(x) \leq 0 \iff f'(x) \leq 1$, and from the graph $f'(x) \leq 1$ for $0 \leq x \leq 5$, the graph of g is decreasing on the interval $0 \leq x \leq 5$ (because $g'(x) \leq 0$ there). Points: (1) correct $g'(x) = f'(x) - 1$ (or equivalent); (2) correct interval $[0, 5]$ with reason.

Solution 3

(d) Mark scheme

- g is continuous, $g'(x) < 0$ for $0 < x < 5$, and $g'(x) > 0$ for $5 < x < 7$, so the absolute minimum occurs at $x = 5$: $g(5) = f(5) - 5 = \frac{7}{2} - 5 = -\frac{3}{2}$. The absolute minimum value of g on $[0, 7]$ is $-\frac{3}{2}$. (Alternative: a Candidates Test comparing $g(0) = 3 + 2\pi$, $g(5) = -3/2$, $g(7) = -1/2$ gives the same result.) Points: (1) considers $g'(x) = 0$, isolating $x = 5$ as the only critical number on $(0, 7)$; (2) correct answer $-3/2$ with justification.

Question 4

2021 ▪ Q4

[Graph of f on a grid, x-axis from -4 to 6, y-axis from -4 to 6, gridlines every 2 units on the y-axis and every 2 units on the x-axis; the graph consists of four connected line segments: from $(-4, 0)$ up to $(-2, 6)$, then down to $(0, 4)$, then down to $(2, -4)$, then up to $(6, 0)$. Labeled "Graph of f".]

Let f be a continuous function defined on the closed interval $-4 \leq x \leq 6$. The graph of f, consisting of four line segments, is shown above. Let G be the function defined by $G(x) = \displaystyle\int_0^x f(t)\,dt$.

(a) On what open intervals is the graph of G concave up? Give a reason for your answer.

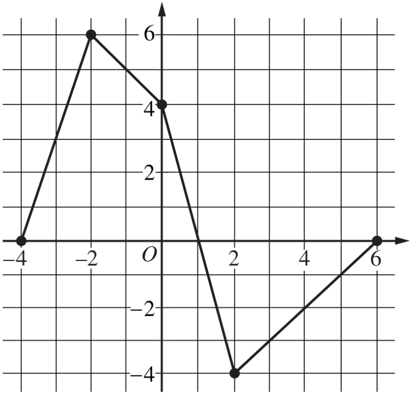
(b) Let P be the function defined by $P(x) = G(x) \cdot f(x)$. Find $P'(3)$.

Question 4

(c) Find $\lim_{x\to 2} \dfrac{G(x)}{x^2 - 2x}$.

(d) Find the average rate of change of G on the interval $[-4, 2]$. Does the Mean Value Theorem guarantee a value c, $-4 < c < 2$, for which $G'(c)$ is equal to this average rate of change? Justify your answer.

Question 4



Graph of f

Solution 4

(a)

Mark scheme

- $G'(x) = f(x)$ (the "global point," worth 1 point, earnable anywhere in parts (a)-(c) by writing $G' = f$, $G'(x) = f(x)$, $G''(x) = f'(x)$, $G'(3) = f(3)$, or $G'(2) = f(2)$). The graph of G is concave up for $-4 < x < -2$ and $2 < x < 6$, because $G' = f$ is increasing on these intervals (intervals may include one or both endpoints). Point (part a, 1 point): correct intervals with the reason that $G' = f$ is increasing there. Point (global, 1 point, folded into this leaf): correctly stating $G'(x) = f(x)$ anywhere in the response.

Solution 4

(b)

Mark scheme

- $P(x) = G(x)f(x) \Rightarrow P'(x) = G(x)f'(x) + f(x)G'(x)$. Substituting $G(3) = \int_0^3 f(t)\,dt = -3.5$ and $G'(3) = f(3) = -3$ (and $f'(3) = 1$, the slope of the line segment from $(2, -4)$ to $(6, 0)$): $P'(3) = -3.5 \cdot 1 + (-3) \cdot (-3) = 5.5$. Point 1: correct application of the product rule in terms of x or evaluated at 3 (once earned, cannot be lost). Point 2: correctly evaluating $G(3) = -3.5$, $G'(3) = -3$, or $f(3) = -3$. Point 3: the final answer 5.5 (needs points 1 and 2 first; simplification not required).

Solution 4

(c) Mark scheme

- $\lim_{x\to 2}(x^2 - 2x) = 0$ and, since G is continuous, $\lim_{x\to 2} G(x) = \int_0^2 f(t)\,dt = 0$, so the limit is the indeterminate form $0/0$. By L'Hospital's Rule, $\lim_{x\to 2} \dfrac{G(x)}{x^2 - 2x} = \lim_{x\to 2} \dfrac{G'(x)}{2x-2} = \lim_{x\to 2} \dfrac{f(x)}{2x-2} = \dfrac{f(2)}{2} = \dfrac{-4}{2} = -2$. Point 1: shows $\lim(x^2 - 2x) = 0$ and $\lim G(x) = 0$, and shows the ratio of the two derivatives $G'(x)$ and $2x - 2$ (may be shown as an evaluation at $x = 2$). Point 2: evaluates the limit correctly with appropriate limit notation, including $\lim_{x\to 2} \frac{G'(x)}{2x-2}$ or $\lim_{x\to 2} \frac{f(x)}{2x-2}$; any linkage error (e.g. writing $\frac{G'(x)}{2x-2} = \frac{f(2)}{2}$ without the limit) forfeits this point.

Solution 4

(d) Mark scheme

- $G(2) = \int_0^2 f(t)\,dt = 0$ and $G(-4) = \int_0^{-4} f(t)\,dt = -16$. Average rate of change $= \dfrac{G(2) - G(-4)}{2 - (-4)} = \dfrac{0 - (-16)}{6} = \dfrac{8}{3}$. Yes — since $G'(x) = f(x)$, G is differentiable on $(-4, 2)$ and continuous on $[-4, 2]$, so the Mean Value Theorem applies and guarantees a value c, $-4 < c < 2$, such that $G'(c) = \dfrac{8}{3}$. Point 1: presents at least a difference and a quotient with correct evaluation (numerical simplification not required, but if simplified must be correct). Point 2: correct conclusion (earnable even with an incorrect/no evaluation of the average rate of change, as long as MVT's conditions and conclusion are stated — an explicit MVT citation is not required).

Solution 4

Question 3

2019 ▪ Q3

The continuous function *f* is defined on the closed interval $-6 \le x \le 5$. The figure below shows a portion of the graph of *f*, consisting of two line segments and a quarter of a circle centered at the point $(5, 3)$. It is known that the point $(3,\ 3-\sqrt{5})$ is on the graph of *f*.

[Graph of *f*; *x*-axis from -2 to 5 (gridlines shown from -2 to 5), *y*-axis from -1 to 3. A line segment goes from a closed point at $(-2, 1)$ down to a closed point at the origin $(0, 0)$. From $(0, 0)$ a line segment rises to a closed point at $(2, 3)$. From $(2, 3)$ a curve (quarter circle centered at $(5, 3)$) decreases and flattens out, passing through $(3, 3-\sqrt{5})$, ending at a closed point at $(5, 0)$ on the *x*-axis. Labeled "Graph of *f*".]

(a) If $\displaystyle\int_{-6}^{5} f(x)\,dx = 7$, find the value of $\displaystyle\int_{-6}^{-2} f(x)\,dx$. Show the work that leads to your answer.

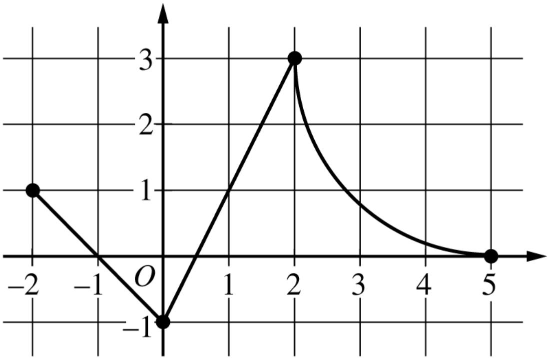
Question 3

(b) Evaluate $\int_3^5 (2f'(x) + 4)\, dx$.

(c) The function g is given by $g(x) = \int_{-2}^{x} f(t)\, dt$. Find the absolute maximum value of g on the interval $-2 \le x \le 5$. Justify your answer.

(d) Find $\lim_{x \to 1} \dfrac{10^x - 3f'(x)}{f(x) - \arctan x}$.

Question 3



Graph of f

Solution 3

(a) Mark scheme

- $\int_{-6}^{5} f(x)\,dx = \int_{-6}^{-2} f(x)\,dx + \int_{-2}^{5} f(x)\,dx$. Given $\int_{-6}^{5} f(x)\,dx = 7$ and (from the given graph of f) $\int_{-2}^{5} f(x)\,dx = 2 + \left(9 - \frac{9\pi}{4}\right) = 11 - \frac{9\pi}{4}$, so $7 = \int_{-6}^{-2} f(x)\,dx + \left(11 - \frac{9\pi}{4}\right)$, giving $\int_{-6}^{-2} f(x)\,dx = 7 - \left(11 - \frac{9\pi}{4}\right) = \frac{9\pi}{4} - 4$. Points: 1 for splitting the integral correctly ($\int_{-6}^{5} = \int_{-6}^{-2} + \int_{-2}^{5}$), 1 for the correct value of $\int_{-2}^{5} f(x)\,dx$, 1 for the final answer $\frac{9\pi}{4} - 4$.

Solution 3

(b)

Mark scheme

- $\int_3^5 (2f'(x)+4)\,dx = 2\int_3^5 f'(x)\,dx + \int_3^5 4\,dx = 2(f(5)-f(3)) + 4(5-3) = 2\left(0-(3-\sqrt{5})\right)+8 = 2(-3+\sqrt{5})+8 = 2+2\sqrt{5}$. Points: 1 for using the Fundamental Theorem of Calculus (i.e. $2(f(5)-f(3))$ term), 1 for the final answer $2+2\sqrt{5}$.

Solution 3

(c) Mark scheme

- $g(x) = \int_{-2}^{x} f(t)\,dt$, so $g'(x) = f(x)$. $g'(x) = 0$ at $x = -1,\ x = \frac{1}{2},\ x = 5$ (the critical points/endpoints from the given graph of f on $[-2, 5]$). Evaluating g at the candidates: $g(-2) = 0$, $g(-1) = \frac{1}{2}$, $g(\frac{1}{2}) = -\frac{1}{4}$, $g(5) = 11 - \frac{9\pi}{4}$. On $-2 \leq x \leq 5$, the absolute maximum value of g is $g(5) = 11 - \frac{9\pi}{4}$. Points: 1 for $g'(x) = f(x)$, 1 for identifying $x = -1$ as a candidate (local max, since f changes from positive to negative there), 1 for the correct answer $g(5) = 11 - \frac{9\pi}{4}$ with justification (candidates test/table).

Solution 3

(d) Mark scheme

- $\lim_{x\to 1} \dfrac{10^x - 3f'(x)}{f(x) - \arctan x} = \dfrac{10^1 - 3f'(1)}{f(1) - \arctan 1} = \dfrac{10 - 3\cdot 2}{1 - \arctan 1} = \dfrac{4}{1 - \frac{\pi}{4}}$. Points: 1 for the correct final answer, $\dfrac{4}{1-\pi/4}$.

Question 1

2015 ▪ Q1

The rate at which rainwater flows into a drainpipe is modeled by the function R, where $R(t) = 20\sin\left(\frac{t^2}{35}\right)$ cubic feet per hour, t is measured in hours, and $0 \leq t \leq 8$. The pipe is partially blocked, allowing water to drain out the other end of the pipe at a rate modeled by $D(t) = -0.04t^3 + 0.4t^2 + 0.96t$ cubic feet per hour, for $0 \leq t \leq 8$. There are 30 cubic feet of water in the pipe at time $t = 0$.

(a) How many cubic feet of rainwater flow into the pipe during the 8-hour time interval $0 \leq t \leq 8$?

(b) Is the amount of water in the pipe increasing or decreasing at time $t = 3$ hours? Give a reason for your answer.

(c) At what time t, $0 \leq t \leq 8$, is the amount of water in the pipe at a minimum? Justify your answer.

Question 1

(d) The pipe can hold 50 cubic feet of water before overflowing. For $t > 8$, water continues to flow into and out of the pipe at the given rates until the pipe begins to overflow. Write, but do not solve, an equation involving one or more integrals that gives the time w when the pipe will begin to overflow.

Solution 1

(a)

Mark scheme

- $\int$ R(t) dt = 76.570 cubic feet. Points: 1 for the integrand R(t) with correct limits of integration 0 to 8, 1 for the correct numeric answer 76.570.

Solution 1

(b)

Mark scheme

- $R(3) - D(3) = -0.313632 < 0$. Since $R(3) < D(3)$, the amount of water in the pipe is decreasing at $t = 3$ hours. Points: 1 for considering (evaluating) both $R(3)$ and $D(3)$, 1 for the correct answer (decreasing) with the reason $R(3) < D(3)$.

Solution 1

(c)

Mark scheme

- The amount of water in the pipe at time t, $0 \leq t \leq 8$, is $W(t) = 30 + \int [R(x) - D(x)]\, dx$. Setting $R(t) - D(t) = 0$ gives $t = 0$ or $t = 3.271658$. Evaluating W at these candidates and the endpoint: $W(0) = 30$, $W(3.271658) = 27.964561$, $W(8) = 48.543686$. The amount of water in the pipe is a minimum at time $t = 3.272$ (or 3.271) hours. Points: 1 for considering $R(t) - D(t) = 0$, 1 for the correct answer $t \approx 3.271$ or 3.272 hours, 1 for justification (comparing W at the candidate point and endpoints, e.g. via a table, to show it is the minimum).

Solution 1

(d)

Mark scheme

- $30 + \int \hat{}w\ [R(t) - D(t)]\ dt = 50$ (equation only — do not solve). Points: 1 for the correct integral $\int \hat{}w\ [R(t) - D(t)]\ dt$, 1 for the complete equation (starting quantity 30 plus that integral set equal to 50).