Past-paper walk-through

Determining Concavity of Functions over Their Domains ▪ 5.6

eXercise

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Questions in this set

- 2019 Q1
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Reading the mark scheme

- **B1** a mark that stands on its own – the point is either made or not.
- **M1** a *method* mark: the right approach, even if the number is wrong.
- **A1** an *answer* mark, and it usually needs its M mark first.
- **C1** a working mark on the way to the answer.
- // separates answers the examiner accepts equally.
- **ecf** = error carried forward: a later part is marked on your own earlier answer.
- **owtte** = "or words to that effect" – the wording need not match.

Question 3

2026 ▪ Q3

A pie is taken from a hot oven and put on a table. The internal temperature of the pie at time t minutes can be modeled by the function H that satisfies the differential equation $\frac{dH}{dt} = -\frac{1}{15}(H - 20)$, where $H(t)$ is measured in degrees Celsius and $H(0) = 75$. For $t > 0$, it is known that $20 < H(t) < 75$.

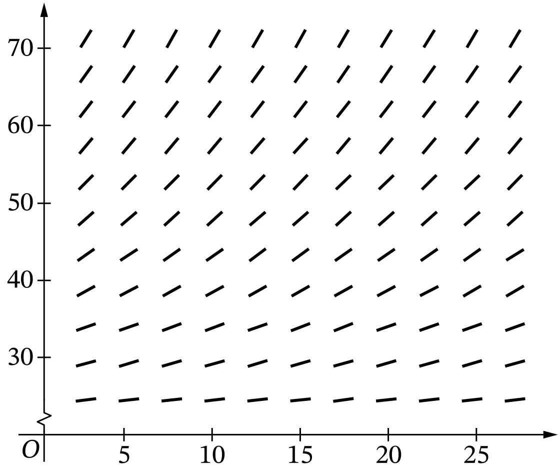
(a) Explain why the following could not be a slope field for the differential equation $\frac{dH}{dt} = -\frac{1}{15}(H - 20)$.

[Slope field diagram: H-axis (vertical) from 30 to 70+ in increments of 10, t-axis (horizontal) from 0 to 25+ in increments of 5. At every grid point across the full range of t shown, the line segments have the same positive slope for a given horizontal row and the slopes appear to depend only on t (segments in each row are parallel and tilt

Question 3

more steeply positive at greater height H, but they do not appear to flatten toward zero as $H \to 20$ nor do the segments vary with t within a row as required) — segments are drawn as short parallel dashes tilting up-and-to-the-right throughout, uniformly across each horizontal band of H values, without change as t increases.]

Question 3



Question 3

2026 ▪ Q3

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(b) Find the slope of the line tangent to the graph of H at time $t = 0$. Show the work that leads to your answer.

Question 3

2026 ▪ Q3

A pie is taken from a hot oven and put on a table. The internal temperature of the pie at time t minutes can be modeled by the function H that satisfies the differential equation $\frac{dH}{dt} = -\frac{1}{15}(H - 20)$, where $H(t)$ is measured in degrees Celsius and $H(0) = 75$. For $t > 0$, it is known that $20 < H(t) < 75$.

(c) It can be shown that $\frac{d^2H}{dt^2} = \frac{1}{225}(H - 20)$. The line tangent to the graph of H at time $t = 0$ is used to approximate $H(5)$, the internal temperature of the pie at time $t = 5$. Is this approximation an overestimate or an underestimate for the actual value of $H(5)$? Give a reason for your answer.

Question 3

2026 ▪ Q3

A pie is taken from a hot oven and put on a table. The internal temperature of the pie at time t minutes can be modeled by the function H that satisfies the differential equation $\frac{dH}{dt} = -\frac{1}{15}(H - 20)$, where $H(t)$ is measured in degrees Celsius and $H(0) = 75$. For $t > 0$, it is known that $20 < H(t) < 75$.

(d) Use separation of variables to find an expression for $H(t)$, the particular solution to the given differential equation with initial condition $H(0) = 75$.

Question 4

2026 ▪ Q4

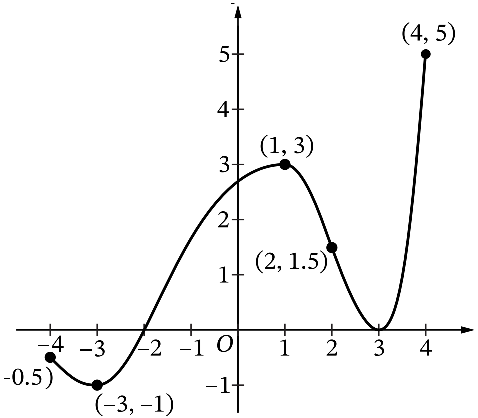
Let f be a twice-differentiable function on the closed interval $[-4, 4]$ with $f(2) = 3$. The graph of f', the derivative of f, is shown.
[Graph of f': x-axis from -4 to 4, y-axis from -1 to 5. The curve passes through labeled points $(-4, -0.5)$, a local minimum at $(-3, -1)$, rises through the x-axis between $x = -2$ and $x = -1$, continues rising to a local maximum at $(1, 3)$, then decreases through $(2, 1.5)$, crosses the x-axis at $x = 3$ (touching down to a minimum of 0 at $x = 3$), then rises sharply to $(4, 5)$.]
(a) For $x > 0$, the function g is defined by $g(x) = f(x) - \ln x$. Find $g'(2)$. Show the work that leads to your answer.
(b) Find all values of x on the open interval $0 < x < 3$ at which the graph of f has a point of inflection. Give a reason for your answer.

Question 4

(c) For $-4 \leq x \leq 4$, on what open intervals, if any, is the graph of f both increasing and concave down? Give a reason for your answer.

(d) For $-4 \leq x \leq 4$, find the value of x at which f has an absolute minimum and the value of x at which f has an absolute maximum. Give reasons for your answers.

Question 4



Graph of f'

Question 1

2024 ▪ Q1

The temperature of coffee in a cup at time t minutes is modeled by a decreasing differentiable function C, where $C(t)$ is measured in degrees Celsius. For $0 \leq t \leq 12$, selected values of $C(t)$ are given in the table shown.

t (minutes)	0	3	7	12
$C(t)$ (degrees Celsius)	100	85	69	55

(a) Approximate $C'(5)$ using the average rate of change of C over the interval $3 \leq t \leq 7$. Show the work that leads to your answer and include units of measure.

Question 1

t (minutes)	0	3	7	12
$C(t)$ (degrees Celsius)	100	85	69	55

Question 1

2024 ▪ Q1

The temperature of coffee in a cup at time t minutes is modeled by a decreasing differentiable function C, where $C(t)$ is measured in degrees Celsius. For $0 \leq t \leq 12$, selected values of $C(t)$ are given in the table shown.

t (minutes)	0	3	7	12
$C(t)$ (degrees Celsius)	100	85	69	55

(b) Use a left Riemann sum with the three subintervals indicated by the data in the table to approximate the value of $\int_0^{12} C(t)\,dt$. Interpret the meaning of $\frac{1}{12}\int_0^{12} C(t)\,dt$ in the context of the problem.

Question 1

2024 ▪ Q1

The temperature of coffee in a cup at time t minutes is modeled by a decreasing differentiable function C, where $C(t)$ is measured in degrees Celsius. For $0 \leq t \leq 12$, selected values of $C(t)$ are given in the table shown.

t (minutes)	0	3	7	12
$C(t)$ (degrees Celsius)	100	85	69	55

(c) For $12 \leq t \leq 20$, the rate of change of the temperature of the coffee is modeled by $C'(t) = \dfrac{-24.55e^{0.01t}}{t}$, where $C'(t)$ is measured in degrees Celsius per minute. Find the temperature of the coffee at time $t = 20$. Show the setup for your calculations.

Question 1

2024 ▪ Q1

The temperature of coffee in a cup at time t minutes is modeled by a decreasing differentiable function C, where $C(t)$ is measured in degrees Celsius. For $0 \leq t \leq 12$, selected values of $C(t)$ are given in the table shown.

t (minutes)	0	3	7	12
$C(t)$ (degrees Celsius)	100	85	69	55

(d) For the model defined in part (c), it can be shown that $C''(t) = \dfrac{0.2455e^{0.01t}(100 - t)}{t^2}$. For $12 < t < 20$, determine whether the temperature of the coffee is changing at a decreasing rate or at an increasing rate. Give a reason for your answer.

Solution 1

(a)

Mark scheme

- $C'(5) \approx \dfrac{C(7) - C(3)}{7 - 3} = \dfrac{69 - 85}{4} = -4$ degrees Celsius per minute. 1 point for the difference-quotient setup with supporting work (a difference over a quotient using table values), 1 point for the correct value with units ('degrees Celsius per minute' attached to the numeric value).

Solution 1

(b) Mark scheme

- Left Riemann sum with the three subintervals from the table: $\int_0^{12} C(t)\,dt \approx (3-0)\cdot C(0) + (7-3)\cdot C(3) + (12-7)\cdot C(7) = 3(100) + 4(85) + 5(69) = 300 + 340 + 345 = 985$. Interpretation: $\frac{1}{12}\int_0^{12} C(t)\,dt$ is the average temperature of the coffee (in degrees Celsius) over the interval $t = 0$ to $t = 12$. 1 point for the correct form of the left Riemann sum (widths times left-endpoint values), 1 point for the numeric estimate 985, 1 point for an interpretation that names both 'average temperature' and the time interval.

Solution 1

(c)

Mark scheme

- $C(20) = C(12) + \int_{12}^{20} C'(t)\,dt = 55 - 14.670812 = 40.329188 \approx 40.329$ degrees Celsius. 1 point for setting up the definite integral of $C'(t)$ with correct limits 12 to 20, 1 point for adding the initial condition $C(12) = 55$ (symbolically or numerically), 1 point for the final numeric answer 40.329 (or 40.330) with supporting work shown.

Solution 1

(d) Mark scheme

- Because $C''(t) > 0$ on $12 < t < 20$ (from $C''(t) = \dfrac{0.2455e^{0.01t}(100 - t)}{t^2}$, numerator and denominator both positive on this interval), the rate of change of temperature $C'(t)$ is increasing on this interval — that is, the temperature of the coffee is changing at an increasing rate. The single point requires a correct answer ('increasing rate') together with a reason that references the sign of the second derivative of C (not just evaluation at one point, and not an ambiguous pronoun).

Question 3

2023 ▪ Q3

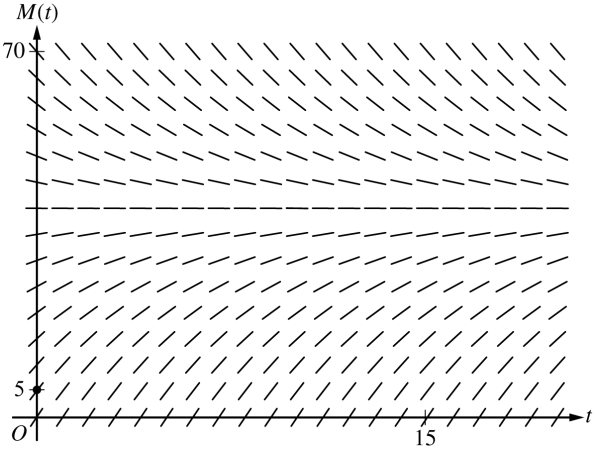
A bottle of milk is taken out of a refrigerator and placed in a pan of hot water to be warmed. The increasing function M models the temperature of the milk at time t, where $M(t)$ is measured in degrees Celsius ($^\circ$C) and t is the number of minutes since the bottle was placed in the pan. M satisfies the differential equation $\dfrac{dM}{dt} = \dfrac{1}{4}(40 - M)$. At time $t = 0$, the temperature of the milk is 5°C. It can be shown that $M(t) < 40$ for all values of t.

(a) A slope field for the differential equation $\dfrac{dM}{dt} = \dfrac{1}{4}(40 - M)$ is shown. Sketch the solution curve through the point $(0, 5)$.
[Slope field diagram: axes labeled $M(t)$ (vertical, with gridlines at 5 and 70 marked) and t (horizontal, with a gridline at 15 marked), origin O. Short line segments are

Question 3

drawn at grid points across the region $0 \leq t \leq 15$-ish, $0 \leq M(t) \leq 70$-ish: segments are nearly horizontal (flat, slightly downward-sloping) near $M(t) = 70$, becoming steeper negative slope moving down toward $M(t) \approx 40$ (segments are horizontal/flat right at $M(t) = 40$), then the slopes become positive and increasingly steep as $M(t)$ decreases from 40 down toward $M(t) = 5$, with the steepest (near-vertical, slope ≈ 1) segments at the bottom of the field near $M(t) = 5$. The point $(0, 5)$ is marked with a star on the vertical axis.]

Question 3



Question 3

2023 ▪ Q3

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(b) Use the line tangent to the graph of M at $t = 0$ to approximate $M(2)$, the temperature of the milk at time $t = 2$ minutes.

Question 3

2023 ▪ Q3

A bottle of milk is taken out of a refrigerator and placed in a pan of hot water to be warmed. The increasing function M models the temperature of the milk at time t, where $M(t)$ is measured in degrees Celsius ($^\circ$C) and t is the number of minutes since the bottle was placed in the pan. M satisfies the differential equation $\dfrac{dM}{dt} = \dfrac{1}{4}(40 - M)$. At time $t = 0$, the temperature of the milk is 5°C. It can be shown that $M(t) < 40$ for all values of t.

(c) Write an expression for $\dfrac{d^2M}{dt^2}$ in terms of M. Use $\dfrac{d^2M}{dt^2}$ to determine whether the approximation from part (b) is an underestimate or an overestimate for the actual value of $M(2)$. Give a reason for your answer.

Question 3

2023 ▪ Q3

A bottle of milk is taken out of a refrigerator and placed in a pan of hot water to be warmed. The increasing function M models the temperature of the milk at time t, where $M(t)$ is measured in degrees Celsius ($^\circ$C) and t is the number of minutes since the bottle was placed in the pan. M satisfies the differential equation $\frac{dM}{dt} = \frac{1}{4}(40 - M)$. At time $t = 0$, the temperature of the milk is 5°C. It can be shown that $M(t) < 40$ for all values of t.

(d) Use separation of variables to find an expression for $M(t)$, the particular solution to the differential equation $\frac{dM}{dt} = \frac{1}{4}(40 - M)$ with initial condition $M(0) = 5$.

Solution 3

(a)

Mark scheme

- The solution curve through (0,5) starts at (0,5), increases and is concave down, and approaches (curving up toward, but always staying below) the flat slope segments at M=40 as t increases across the shown window; it must not cross any drawn slope-field segments (1 pt: curve passes through (0,5), extends reasonably close to the left and right edges of the given rectangle, has no obvious conflicts with the slope lines, and lies entirely below M=40).

Solution 3

(b) Mark scheme

- $\left.\dfrac{dM}{dt}\right|_{t=0} = \dfrac{1}{4}(40-5) = \dfrac{35}{4}$ (1 pt: value of the derivative at t=0). Tangent line: $y = 5 + \dfrac{35}{4}(t-0)$, so $M(2) \approx 5 + \dfrac{35}{4} \cdot 2 = 22.5$; the temperature of the milk at t=2 minutes is approximately 22.5°C (1 pt: approximation using a tangent line through (0,5) with slope 35/4, correctly simplified).

Solution 3

(c) Mark scheme

- $\dfrac{d^2M}{dt^2} = -\dfrac{1}{4}\dfrac{dM}{dt} = -\dfrac{1}{4}\left(\dfrac{1}{4}(40 - M)\right) = -\dfrac{1}{16}(40 - M)$ (1 pt: correct second-derivative expression in terms of M, no subsequent simplification error). Since $M(t) < 40$, $\dfrac{d^2M}{dt^2} < 0$, so the graph of M is concave down; therefore the tangent-line approximation from part (b) is an overestimate (1 pt: overestimate conclusion with a reason based on concavity/decreasing dM/dt).

Solution 3

(d) Mark scheme

- Separate variables: $\dfrac{dM}{40-M} = \dfrac{1}{4}\,dt$ (1 pt: separates variables). Integrate: $\displaystyle\int \frac{dM}{40-M} = \int \frac{1}{4}\,dt \Rightarrow -\ln|40-M| = \frac{1}{4}t + C$ (1 pt: finds antiderivatives). Apply the initial condition M(0)=5: $-\ln|40-5| = 0 + C \Rightarrow C = -\ln 35$; since M(t)<40, $|40-M| = 40-M$, giving $-\ln(40-M) = \dfrac{1}{4}t - \ln 35$, i.e. $\ln(40-M) = -\dfrac{1}{4}t + \ln 35$ (1 pt: constant of integration correctly found and initial condition used). Solve for M: $40 - M = 35e^{-t/4} \Rightarrow M(t) = 40 - 35e^{-t/4}$ (1 pt: solves for M — the final answer must be exactly this closed form, or equivalent).

Question 4

2023 ▪ Q4

The function f is defined on the closed interval $[-2, 8]$ and satisfies $f(2) = 1$. The graph of f', the derivative of f, consists of two line segments and a semicircle, as shown in the figure.
[Graph of f': x-axis from -2 to 8 with gridlines at every integer, y-axis from -2 to 2 with gridlines at every integer. The graph consists of: a line segment from $(-2, 2)$ down to $(0, -2)$; a line segment from $(0, -2)$ up to $(4, 2)$; and a semicircle (bulging downward, below the x-axis) connecting $(4, 2)$ down through the x-axis and back up to $(8, 2)$, dipping to a minimum of about 0 at $x = 6$. Solid dots are marked at $(-2, 2)$, $(0, -2)$, $(4, 2)$, and $(8, 2)$.]
(a) Does f have a relative minimum, a relative maximum, or neither at $x = 6$? Give a reason for your answer.

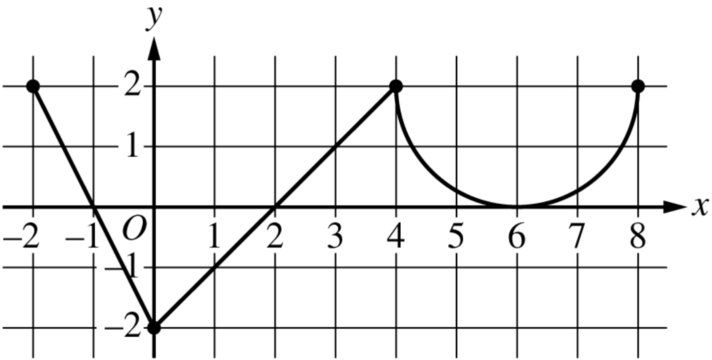
Question 4

(b) On what open intervals, if any, is the graph of f concave down? Give a reason for your answer.

(c) Find the value of $\lim_{x \to 2} \dfrac{6f(x) - 3x}{x^2 - 5x + 6}$, or show that it does not exist. Justify your answer.

(d) Find the absolute minimum value of f on the closed interval $[-2, 8]$. Justify your answer.

Question 4



Graph of f'

Solution 4

(a)

Mark scheme

- $f'(x) > 0$ on (2,6) and $f'(x) > 0$ on (6,8), so f'(x) does not change sign at x=6; therefore f has neither a relative maximum nor a relative minimum at x=6 (1 pt: correct answer — 'neither' — with the reason that f'(x) does not change sign at x=6).

Solution 4

(b)

Mark scheme

- The graph of f is concave down on (-2,0) and (4,6) because f' is decreasing on these intervals (1 pt: correct intervals (-2,0) and (4,6) — with or without endpoints; 1 pt: reason, must correctly discuss the behavior/slopes of f').

Solution 4

(c) Mark scheme

- Because f is differentiable, hence continuous, at x=2, $\lim_{x\to 2} f(x) = f(2) = 1$; so $\lim_{x\to 2}(6f(x) - 3x) = 6(1) - 3(2) = 0$ and $\lim_{x\to 2}(x^2 - 5x + 6) = 0$ (1 pt: both limits of numerator and denominator shown to equal 0; a limit presented as literally equal to 0/0 does not earn this point). Since the limit is of indeterminate form 0/0, L'Hospital's Rule applies (1 pt: uses L'Hospital's Rule — at least one correct derivative shown in the limit of a ratio of derivatives). $\lim_{x\to 2} \dfrac{6f(x) - 3x}{x^2 - 5x + 6} = \lim_{x\to 2} \dfrac{6f'(x) - 3}{2x - 5} = \dfrac{6(0) - 3}{2(2) - 5} = \dfrac{-3}{-1} = 3$ (1 pt: correct answer 3 with supporting work).

Solution 4

(d)

Mark scheme

- $f'(x) = 0 \Rightarrow x = -1, 2, 6$ (1 pt: considers f'(x)=0 — merely listing the zeros of f' without stating f'=0 is not sufficient). Since f is continuous on [-2,8], the candidates for the location of an absolute minimum are x=-2,-1,2,6,8, with $f(-2) = 3$, $f(-1) = 4$, $f(2) = 1$, $f(6) = 7 - \pi$, $f(8) = 11 - 2\pi$ (1 pt: justification — evaluate f at all critical points and both endpoints, or validly eliminate some by argument). The absolute minimum value of f is $f(2) = 1$ (1 pt: answer — must state the minimum VALUE 1, not merely its location x=2).

Question 5

2023 ▪ Q5

The functions f and g are twice differentiable. The table shown gives values of the functions and their first derivatives at selected values of x.

x	0	2	4	7
$f(x)$	10	7	4	5
$f'(x)$	$\frac{3}{2}$	-8	3	6
$g(x)$	1	2	-3	0
$g'(x)$	5	4	2	8

(a) Let h be the function defined by $h(x) = f(g(x))$. Find $h'(7)$. Show the work that leads to your answer.

(b) Let k be a differentiable function such that $k'(x) = (f(x))^2 \cdot g(x)$. Is the graph of k concave up or concave down at the point where $x = 4$? Give a reason for your answer.

(c) Let m be the function defined by $m(x) = 5x^3 + \int_0^x f'(t)\,dt$. Find $m(2)$. Show the work that leads to your answer.

Question 5

(d) Is the function m defined in part (c) increasing, decreasing, or neither at $x = 2$? Justify your answer.

Question 5

x	0	2	4	7
$f(x)$	10	7	4	5
$f'(x)$	$\frac{3}{2}$	-8	3	6
$g(x)$	1	2	-3	0
$g'(x)$	5	4	2	8

Solution 5

(a)

Mark scheme

- $h'(x) = f'(g(x)) \cdot g'(x)$ (1 pt: chain rule).
  $h'(7) = f'(g(7)) \cdot g'(7) = f'(0) \cdot 8 = \frac{3}{2} \cdot 8 = 12$ (1 pt: correct answer 12, using the table values g(7)=0, f'(0)=3/2, g'(7)=8).

Solution 5

(b)

Mark scheme

- $k''(x) = 2f(x)f'(x)g(x) + (f(x))^2 g'(x)$ (1 pt: product/chain rule applied to k'(x)=(f(x))^2 g(x)). $k''(4) = 2f(4)f'(4)g(4) + (f(4))^2 g'(4) = 2(4)(3)(-3) + (4)^2(2) = -72 + 32 = -40$ (1 pt: k''(4)=-40 with supporting work using the table values f(4)=4, f'(4)=3, g(4)=-3, g'(4)=2). Since k''(4)=-40<0 and k'' is continuous, the graph of k is concave down at x=4 (1 pt: answer with reason, consistent with the declared nonzero value of k''(4)).

Solution 5

(c)

Mark scheme

- By the Fundamental Theorem of Calculus, $m(2) = 5(2)^3 + \int_0^2 f'(t)\,dt = 40 + (f(2) - f(0)) = 40 + (7 - 10) = 37$ (1 pt: answer 37 with supporting work equivalent to $5 \cdot 8 + (f(2) - f(0))$).

Solution 5

(d)

Mark scheme

- $m'(x) = 15x^2 + f'(x)$ (1 pt: considers m'(x)).
  $m'(2) = 15(2)^2 + f'(2) = 60 + (-8) = 52$ (1 pt: m'(2) with supporting work, using f'(2)=-8 from the table; an unsupported m'(2)=52 does not earn this point). Since $m'(2) = 52 > 0$, the graph of m is increasing at x=2 (1 pt: answer with justification, consistent with the declared value of m'(2)).

Question 3

2022 ▪ Q3

[Graph titled "Graph of f'": a coordinate plane with x-axis from 0 to 7 (integer gridlines at $1, 2, 3, 4, 5, 6, 7$) and y-axis from -2 to 2 (gridlines at $-2, -1, 1, 2$). The graph consists of a semicircle below the x-axis from $x = 0$ to $x = 4$, dipping down to a minimum of about -2 near $x = 2$, returning to 0 at $x = 4$. From $x = 4$ to $x = 6$ a line segment rises from $(4, 0)$ to the marked point $(6, 2)$. From $x = 6$ to $x = 7$ a line segment falls from $(6, 2)$ to the marked point $(7, 1)$.]

Let f be a differentiable function with $f(4) = 3$. On the interval $0 \leq x \leq 7$, the graph of f', the derivative of f, consists of a semicircle and two line segments, as shown in the figure above.

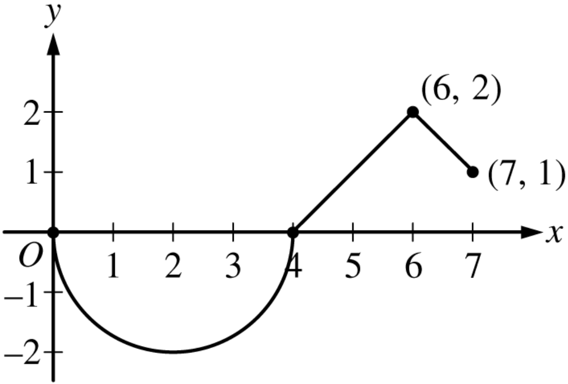
(a) Find $f(0)$ and $f(5)$.

(b) Find the x-coordinates of all points of inflection of the graph of f for $0 < x < 7$. Justify your answer.

Question 3

(c) Let g be the function defined by $g(x) = f(x) - x$. On what intervals, if any, is g decreasing for $0 \leq x \leq 7$? Show the analysis that leads to your answer.
(d) For the function g defined in part (c), find the absolute minimum value on the interval $0 \leq x \leq 7$. Justify your answer.

Question 3



Graph of f'

Solution 3

(a)

Mark scheme

- $f(0) = f(4) + \int_4^0 f'(x)\,dx = 3 - \int_0^4 f'(x)\,dx = 3 + 2\pi$ (the semicircle over $[0, 4]$ has radius 2 and lies below the axis, so $\int_0^4 f'(x)\,dx = -2\pi$). $f(5) = f(4) + \int_4^5 f'(x)\,dx = 3 + \frac{1}{2} = \frac{7}{2}$. Points: (1) area of either region ($\int_0^4 f'$ or $\int_4^5 f'$); (2) correct $f(0) = 3 + 2\pi$; (3) correct $f(5) = 7/2$.

Solution 3

(b)

Mark scheme

- The graph of f has points of inflection at $x = 2$ and $x = 6$, because f' changes from decreasing to increasing at $x = 2$ (the bottom of the semicircle) and from increasing to decreasing at $x = 6$ (the corner/peak of the graph of f' at $(6, 2)$). Points: (1) states both $x = 2$ and $x = 6$; (2) correct justification tied to the graph of f' (e.g., discussing where the slope/behavior of f' changes, or citing $x = 2, 6$ as locations of local extrema on the graph of f').

Solution 3

(c)

Mark scheme

- $g(x) = f(x) - x \Rightarrow g'(x) = f'(x) - 1$. Since $g'(x) \leq 0 \iff f'(x) \leq 1$, and from the graph $f'(x) \leq 1$ for $0 \leq x \leq 5$, the graph of g is decreasing on the interval $0 \leq x \leq 5$ (because $g'(x) \leq 0$ there). Points: (1) correct $g'(x) = f'(x) - 1$ (or equivalent); (2) correct interval $[0, 5]$ with reason.

Solution 3

(d) Mark scheme

- g is continuous, $g'(x) < 0$ for $0 < x < 5$, and $g'(x) > 0$ for $5 < x < 7$, so the absolute minimum occurs at $x = 5$: $g(5) = f(5) - 5 = \frac{7}{2} - 5 = -\frac{3}{2}$. The absolute minimum value of g on $[0, 7]$ is $-\frac{3}{2}$. (Alternative: a Candidates Test comparing $g(0) = 3 + 2\pi$, $g(5) = -3/2$, $g(7) = -1/2$ gives the same result.) Points: (1) considers $g'(x) = 0$, isolating $x = 5$ as the only critical number on $(0, 7)$; (2) correct answer $-3/2$ with justification.

Question 5

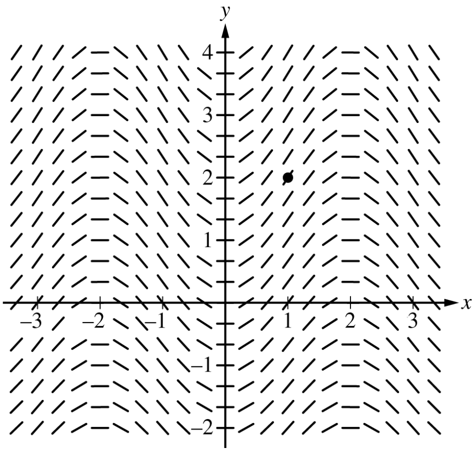
2022 ▪ Q5

Consider the differential equation $\dfrac{dy}{dx} = \dfrac{1}{2}\sin\left(\dfrac{\pi}{2}x\right)\sqrt{y+7}$. Let $y = f(x)$ be the particular solution to the differential equation with the initial condition $f(1) = 2$. The function f is defined for all real numbers.

(a) A portion of the slope field for the differential equation is given below. Sketch the solution curve through the point $(1, 2)$.

[Graph: a slope field on the region $-3 \leq x \leq 3$, $-2 \leq y \leq 4$ (integer gridlines on both axes). Short line segments at each grid point show the local slope; segments in the left portion (roughly $x < 0$) slant like "/" (positive slope, forward slashes), and segments in the right portion (roughly $x > 0$) slant like "(negative slope, back slashes), with a marked point at $(1, 2)$ where the solution curve should be sketched.]

Question 5



Question 5

2022 ▪ Q5

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(b) Write an equation for the line tangent to the solution curve in part (a) at the point $(1, 2)$. Use the equation to approximate $f(0.8)$.

Question 5

2022 ▪ Q5

Consider the differential equation $\dfrac{dy}{dx} = \dfrac{1}{2}\sin\left(\dfrac{\pi}{2}x\right)\sqrt{y+7}$. Let $y = f(x)$ be the particular solution to the differential equation with the initial condition $f(1) = 2$. The function f is defined for all real numbers.

(c) It is known that $f''(x) > 0$ for $-1 \leq x \leq 1$. Is the approximation found in part (b) an overestimate or an underestimate for $f(0.8)$? Give a reason for your answer.

Question 5

2022 ▪ Q5

Consider the differential equation $\frac{dy}{dx} = \frac{1}{2}\sin\left(\frac{\pi}{2}x\right)\sqrt{y+7}$. Let $y = f(x)$ be the particular solution to the differential equation with the initial condition $f(1) = 2$. The function f is defined for all real numbers.

(d) Use separation of variables to find $y = f(x)$, the particular solution to the differential equation

$$\frac{dy}{dx} = \frac{1}{2}\sin\left(\frac{\pi}{2}x\right)\sqrt{y+7}$$

with the initial condition $f(1) = 2$.

Solution 5

(a)

Mark scheme

- The solution curve must pass through $(1, 2)$, follow the given slope field, satisfy $f(x) > 0$ at every point on the curve, extend reasonably close to the left and right edges of the given square, and have any local maximum/minimum occur only where the slope field shows horizontal segments. This is a graphical sketch, not an algebraic answer; 1 point is earned for a curve satisfying these properties.

Solution 5

(b)

Mark scheme

- $\left.\dfrac{dy}{dx}\right|_{(x,y)=(1,2)} = \dfrac{1}{2}\sin\left(\dfrac{\pi}{2}\right)\sqrt{2+7} = \dfrac{1}{2}\cdot 1\cdot 3 = \dfrac{3}{2}$. Tangent line: $y = 2 + \dfrac{3}{2}(x-1)$. Approximation: $f(0.8) \approx 2 + \dfrac{3}{2}(0.8-1) = 1.7$. Points: (1) correct tangent line equation (any equivalent form); (2) correct approximation $f(0.8) \approx 1.7$.

Solution 5

(c)

Mark scheme

- Because $f''(x) > 0$ on $-1 \leq x \leq 1$, f is concave up there, so the tangent line lies below the graph of $y = f(x)$ at $x = 0.8$, and the approximation for $f(0.8)$ found in part (b) is an underestimate. 1 point for the correct answer with a reason that references $f''(x) > 0$, f' increasing, or f concave up.

Solution 5

(d) Mark scheme

- $\dfrac{dy}{dx} = \dfrac{1}{2}\sin\left(\dfrac{\pi}{2}x\right)\sqrt{y+7} \Rightarrow \displaystyle\int \frac{dy}{\sqrt{y+7}} = \int \frac{1}{2}\sin\left(\frac{\pi}{2}x\right)dx$. Integrating: $2\sqrt{y+7} = -\dfrac{1}{\pi}\cos\left(\dfrac{\pi}{2}x\right) + C$. Using $f(1) = 2$: $2\sqrt{2+7} = -\dfrac{1}{\pi}\cos\left(\dfrac{\pi}{2}\right) + C \Rightarrow 6 = 0 + C \Rightarrow C = 6$. So $\sqrt{y+7} = 3 - \dfrac{1}{2\pi}\cos\left(\dfrac{\pi}{2}x\right)$, and $y = \left(3 - \dfrac{1}{2\pi}\cos\left(\dfrac{\pi}{2}x\right)\right)^2 - 7$. Points: (1) correct separation of variables; (2) one correct antiderivative; (3) the other correct antiderivative; (4) correct constant of integration found using the initial condition; (5) correctly solves for y.

Question 4

2021 ▪ Q4

[Graph of f on a grid, x-axis from -4 to 6, y-axis from -4 to 6, gridlines every 2 units on the y-axis and every 2 units on the x-axis; the graph consists of four connected line segments: from $(-4, 0)$ up to $(-2, 6)$, then down to $(0, 4)$, then down to $(2, -4)$, then up to $(6, 0)$. Labeled "Graph of f".]

Let f be a continuous function defined on the closed interval $-4 \leq x \leq 6$. The graph of f, consisting of four line segments, is shown above. Let G be the function defined by $G(x) = \displaystyle\int_0^x f(t)\,dt$.

(a) On what open intervals is the graph of G concave up? Give a reason for your answer.

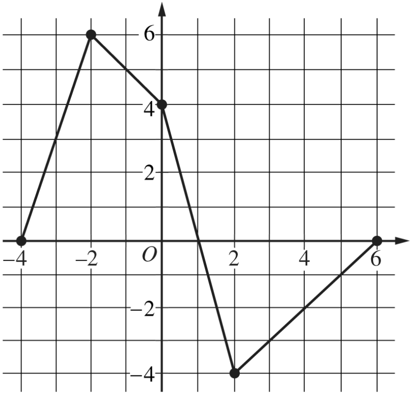
(b) Let P be the function defined by $P(x) = G(x) \cdot f(x)$. Find $P'(3)$.

Question 4

(c) Find $\lim\limits_{x \to 2} \dfrac{G(x)}{x^2 - 2x}$.

(d) Find the average rate of change of G on the interval $[-4, 2]$. Does the Mean Value Theorem guarantee a value c, $-4 < c < 2$, for which $G'(c)$ is equal to this average rate of change? Justify your answer.

Question 4



Graph of f

Solution 4

(a)

Mark scheme

- $G'(x) = f(x)$ (the "global point," worth 1 point, earnable anywhere in parts (a)-(c) by writing $G' = f$, $G'(x) = f(x)$, $G''(x) = f'(x)$, $G'(3) = f(3)$, or $G'(2) = f(2)$). The graph of G is concave up for $-4 < x < -2$ and $2 < x < 6$, because $G' = f$ is increasing on these intervals (intervals may include one or both endpoints). Point (part a, 1 point): correct intervals with the reason that $G' = f$ is increasing there. Point (global, 1 point, folded into this leaf): correctly stating $G'(x) = f(x)$ anywhere in the response.

Solution 4

(b)

Mark scheme

- $P(x) = G(x)f(x) \Rightarrow P'(x) = G(x)f'(x) + f(x)G'(x)$. Substituting $G(3) = \int_0^3 f(t)\,dt = -3.5$ and $G'(3) = f(3) = -3$ (and $f'(3) = 1$, the slope of the line segment from $(2, -4)$ to $(6, 0)$): $P'(3) = -3.5 \cdot 1 + (-3) \cdot (-3) = 5.5$. Point 1: correct application of the product rule in terms of x or evaluated at 3 (once earned, cannot be lost). Point 2: correctly evaluating $G(3) = -3.5$, $G'(3) = -3$, or $f(3) = -3$. Point 3: the final answer 5.5 (needs points 1 and 2 first; simplification not required).

Solution 4

(c) Mark scheme

- $\lim_{x\to 2}(x^2 - 2x) = 0$ and, since G is continuous, $\lim_{x\to 2} G(x) = \int_0^2 f(t)\,dt = 0$, so the limit is the indeterminate form $0/0$. By L'Hospital's Rule, $\lim_{x\to 2} \dfrac{G(x)}{x^2 - 2x} = \lim_{x\to 2} \dfrac{G'(x)}{2x - 2} = \lim_{x\to 2} \dfrac{f(x)}{2x - 2} = \dfrac{f(2)}{2} = \dfrac{-4}{2} = -2$. Point 1: shows $\lim(x^2 - 2x) = 0$ and $\lim G(x) = 0$, and shows the ratio of the two derivatives $G'(x)$ and $2x - 2$ (may be shown as an evaluation at $x = 2$). Point 2: evaluates the limit correctly with appropriate limit notation, including $\lim_{x\to 2} \frac{G'(x)}{2x-2}$ or $\lim_{x\to 2} \frac{f(x)}{2x-2}$; any linkage error (e.g. writing $\frac{G'(x)}{2x-2} = \frac{f(2)}{2}$ without the limit) forfeits this point.

Solution 4

(d) Mark scheme

- $G(2) = \int_0^2 f(t)\,dt = 0$ and $G(-4) = \int_0^{-4} f(t)\,dt = -16$. Average rate of change $= \dfrac{G(2) - G(-4)}{2 - (-4)} = \dfrac{0 - (-16)}{6} = \dfrac{8}{3}$. Yes — since $G'(x) = f(x)$, G is differentiable on $(-4, 2)$ and continuous on $[-4, 2]$, so the Mean Value Theorem applies and guarantees a value c, $-4 < c < 2$, such that $G'(c) = \dfrac{8}{3}$. Point 1: presents at least a difference and a quotient with correct evaluation (numerical simplification not required, but if simplified must be correct). Point 2: correct conclusion (earnable even with an incorrect/no evaluation of the average rate of change, as long as MVT's conditions and conclusion are stated — an explicit MVT citation is not required).

Solution 4

Question 6

2021 ▪ Q6

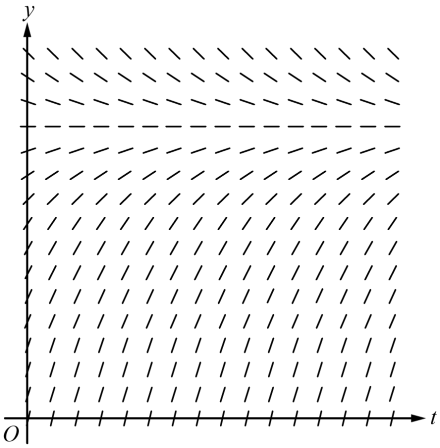
A medication is administered to a patient. The amount, in milligrams, of the medication in the patient at time t hours is modeled by a function $y = A(t)$ that satisfies the differential equation $\frac{dy}{dt} = \frac{12 - y}{3}$. At time $t = 0$ hours, there are 0 milligrams of the medication in the patient.

(a) A portion of the slope field for the differential equation $\frac{dy}{dt} = \frac{12 - y}{3}$ is given below. Sketch the solution curve through the point $(0, 0)$.
[Slope field diagram, t-axis horizontal from 0 rightward, y-axis vertical; short line segments drawn at grid points showing slope direction: segments near the top of the grid slope downward (negative slope, since $y > 12$ there), a horizontal row of flat dashes at $y = 12$ (zero slope), and segments below slope upward (positive slope, since

Question 6

$y < 12$), with segments closer to $y = 0$ sloping steeply upward and flattening as y approaches 12 across each column.]

Question 6



Question 6

2021 ▪ Q6

A medication is administered to a patient. The amount, in milligrams, of the medication in the patient at time t hours is modeled by a function $y = A(t)$ that satisfies the differential equation $\dfrac{dy}{dt} = \dfrac{12 - y}{3}$. At time $t = 0$ hours, there are 0 milligrams of the medication in the patient.

(b) Using correct units, interpret the statement $\lim_{t \to \infty} A(t) = 12$ in the context of this problem.

Question 6

2021 ▪ Q6

A medication is administered to a patient. The amount, in milligrams, of the medication in the patient at time t hours is modeled by a function $y = A(t)$ that satisfies the differential equation $\frac{dy}{dt} = \frac{12-y}{3}$. At time $t = 0$ hours, there are 0 milligrams of the medication in the patient.

(c) Use separation of variables to find $y = A(t)$, the particular solution to the differential equation $\frac{dy}{dt} = \frac{12-y}{3}$ with initial condition $A(0) = 0$.

Question 6

2021 ▪ Q6

A medication is administered to a patient. The amount, in milligrams, of the medication in the patient at time t hours is modeled by a function $y = A(t)$ that satisfies the differential equation $\frac{dy}{dt} = \frac{12 - y}{3}$. At time $t = 0$ hours, there are 0 milligrams of the medication in the patient.

(d) A different procedure is used to administer the medication to a second patient. The amount, in milligrams, of the medication in the second patient at time t hours is modeled by a function $y = B(t)$ that satisfies the differential equation $\frac{dy}{dt} = 3 - \frac{y}{t+2}$. At time $t = 1$ hour, there are 2.5 milligrams of the medication in the second patient. Is the rate of change of the amount of medication in the second patient increasing or decreasing at time $t = 1$? Give a reason for your answer.

Solution 6

(a) Mark scheme

- Sketch the solution curve to $\dfrac{dy}{dt} = \dfrac{12 - y}{3}$ through $(0, 0)$: the curve must start at $(0, 0)$, be generally increasing, be concave down throughout, and approach the horizontal asymptote $y = 12$ from below as t increases (following the slope-field segments, which are positive below $y = 12$, flat at $y = 12$, and negative above $y = 12$). Point 1: earned only if a single such curve is drawn passing through $(0, 0)$ with the correct increasing/concave-down/asymptotic shape; not earned if two or more solution curves are presented.

Solution 6

(b)

Mark scheme

- Over time, the amount of medication in the patient approaches 12 milligrams. Point 1: the interpretation must include "medication in the patient," "approaches 12," and units (milligrams), or their equivalents.

Solution 6

(c) Mark scheme

- Separate variables: $\dfrac{dy}{dt} = \dfrac{12-y}{3} \Rightarrow \dfrac{dy}{12-y} = \dfrac{dt}{3}$. Integrate: $\displaystyle\int \frac{dy}{12-y} = \int \frac{dt}{3} \Rightarrow -\ln|12-y| = \frac{t}{3} + C$. Solve for y: $\ln|12-y| = -\dfrac{t}{3} - C \Rightarrow |12-y| = e^{-t/3-C} \Rightarrow y = 12 + Ke^{-t/3}$. Apply $A(0) = 0$: $0 = 12 + K \Rightarrow K = -12$. Final: $y = A(t) = 12 - 12e^{-t/3}$. Point 1: correct separation of variables ($dy/(12-y) = dt/3$; a "bad separation" like $dy/(12-y) = 3\,dt$ does not earn this point but the response stays eligible for points 2-3). Point 2: correct antiderivatives on both sides (absolute value bars not required). Point 3: correctly introduces and solves for the constant of integration

Solution 6

using the initial condition $A(0) = 0$ (earned consecutively after point 2). Point 4: solves explicitly for $y = A(t) = 12 - 12e^{-t/3}$ (needs points 1-3 in the standard path).

Solution 6

(d) Mark scheme

- $\dfrac{dy}{dt} = 3 - \dfrac{y}{t+2} \Rightarrow \dfrac{d^2y}{dt^2} = -\dfrac{\frac{dy}{dt}(t+2) - y}{(t+2)^2}$ (quotient rule). At $t = 1$: $B'(1) = 3 - \dfrac{B(1)}{3} = 3 - \dfrac{2.5}{3} = \dfrac{6.5}{3}$. Then $B''(1) = -\dfrac{B'(1)\cdot 3 - B(1)}{3^2} = -\dfrac{6.5 - 2.5}{9} = -\dfrac{4}{9} < 0$. The rate of change of the amount of medication in the second patient is DECREASING at $t = 1$ because $B''(1) < 0$ and d^2y/dt^2 is continuous on an interval containing $t = 1$. Point 1: correctly applies the quotient rule to $y/(t+2)$ (or product rule to $y(t+2)^{-1}$); errors differentiating the constant 3 or mishandling the sign forfeit this point. Point 2: states $B''(1) < 0$ with a correct value/sign — cannot be earned unless

Solution 6

the second derivative itself is correct. Point 3: consistent conclusion (decreasing) based on the declared value/sign of $B''(1)$; only needs an attempt at the quotient/product rule to be eligible.

Question 1

2019 ▪ Q1

Fish enter a lake at a rate modeled by the function E given by $E(t) = 20 + 15\sin\left(\frac{\pi t}{6}\right)$. Fish leave the lake at a rate modeled by the function L given by $L(t) = 4 + 2^{0.1t^2}$. Both $E(t)$ and $L(t)$ are measured in fish per hour, and t is measured in hours since midnight $(t = 0)$.

(a) How many fish enter the lake over the 5-hour period from midnight $(t = 0)$ to 5 A.M. $(t = 5)$? Give your answer to the nearest whole number.

(b) What is the average number of fish that leave the lake per hour over the 5-hour period from midnight $(t = 0)$ to 5 A.M. $(t = 5)$?

(c) At what time t, for $0 \leq t \leq 8$, is the greatest number of fish in the lake? Justify your answer.

Question 1

(d) Is the rate of change in the number of fish in the lake increasing or decreasing at 5 A.M. ($t = 5$)? Explain your reasoning.

Solution 1

(a)

Mark scheme

- $\int_0^5 E(t)\,dt = 153.457690$ (calculator value). To the nearest whole number, 153 fish enter the lake from midnight to 5 A.M. Points: 1 for the integral expression, 1 for the correct answer 153.

Solution 1

(b)

Mark scheme

- Average rate $= \dfrac{1}{5-0}\int_0^5 L(t)\,dt = 6.059038$. The average number of fish that leave the lake per hour from midnight to 5 A.M. is 6.059 fish per hour. Points: 1 for the integral expression, 1 for the correct answer 6.059.

Solution 1

(c) Mark scheme

- The rate of change in the number of fish in the lake at time t is $E(t) - L(t)$. Setting $E(t) - L(t) = 0$ gives $t = 6.20356$. Since $E(t) - L(t) > 0$ for $0 \leq t < 6.20356$ and $E(t) - L(t) < 0$ for $6.20356 < t \leq 8$, the number of fish increases then decreases, so the greatest number of fish in the lake is at time $t = 6.204$ (or 6.203). [Alternate: let $A(t) = \int_0^t (E(s) - L(s))\, ds$ be the change in fish since midnight; $A'(t) = E(t) - L(t) = 0$ at $t = C = 6.20356$; compare $A(0) = 0$, $A(C) = 135.01492$, $A(8) = 80.91998$ — the maximum is at $t = C$.] Points: 1 for setting $E(t) - L(t) = 0$ (or the equivalent candidates-test setup), 1 for the answer $t = 6.204$ (or 6.203), 1 for justification (sign analysis or candidates test).

Solution 1

(d)

Mark scheme

- $E'(5) - L'(5) = -10.7228 < 0$. Since the rate of change in the number of fish is $E(t) - L(t)$, its derivative at $t = 5$ is $E'(5) - L'(5)$, which is negative, so the rate of change in the number of fish in the lake is DECREASING at $t = 5$ A.M. Points: 1 for considering $E'(5)$ and $L'(5)$, 1 for the correct answer (decreasing) with explanation.

Question 4

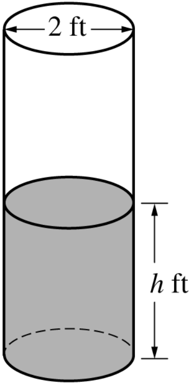
2019 ▪ Q4

A cylindrical barrel with a diameter of 2 feet contains collected rainwater, as shown in the figure below. The water drains out through a valve (not shown) at the bottom of the barrel. The rate of change of the height h of the water in the barrel with respect to time t is modeled by $\dfrac{dh}{dt} = -\dfrac{1}{10}\sqrt{h}$, where h is measured in feet and t is measured in seconds. (The volume V of a cylinder with radius r and height h is $V = \pi r^2 h$.)
[Diagram of a cylindrical barrel with diameter labeled "2 ft" at the top opening; the barrel is partially filled with shaded (gray) water up to a level, with a vertical double-arrow labeled "*h* ft" marking the height of the water from the bottom of the barrel to the water's surface.]
(a) Find the rate of change of the volume of water in the barrel with respect to time when the height of the water is 4 feet. Indicate units of measure.

Question 4

(b) When the height of the water is 3 feet, is the rate of change of the height of the water with respect to time increasing or decreasing? Explain your reasoning.
(c) At time $t = 0$ seconds, the height of the water is 5 feet. Use separation of variables to find an expression for h in terms of t.

Question 4



Solution 4

(a) Mark scheme

- $V = \pi r^2 h = \pi(1)^2 h = \pi h$ (radius fixed at 1 ft), so $\dfrac{dV}{dt} = \pi \dfrac{dh}{dt}$. At $h = 4$, $\dfrac{dh}{dt}\Big|_{h=4} = -\frac{1}{10}\sqrt{4} = -\frac{1}{5}$. So $\dfrac{dV}{dt}\Big|_{h=4} = \pi\left(-\frac{1}{5}\right) = -\dfrac{\pi}{5}$ cubic feet per second.
  Points: 1 for $\frac{dV}{dt} = \pi\frac{dh}{dt}$, 1 for the correct answer with units ($-\pi/5$ cubic feet per second).

Solution 4

(b) Mark scheme

- $\frac{d^2h}{dt^2} = \frac{d}{dh}\left(-\frac{1}{10}\sqrt{h}\right) \cdot \frac{dh}{dt} = -\frac{1}{20\sqrt{h}} \cdot \left(-\frac{1}{10}\sqrt{h}\right) = \frac{1}{200}$. Because $\frac{d^2h}{dt^2} = \frac{1}{200} > 0$ for $h > 0$, the rate of change of the height of the water is INCREASING when the height of the water is 3 feet. Points: 1 for $\frac{d}{dh}\left(-\frac{1}{10}\sqrt{h}\right) = -\frac{1}{20\sqrt{h}}$, 1 for the chain-rule setup $\frac{d^2h}{dt^2} = -\frac{1}{20\sqrt{h}} \cdot \frac{dh}{dt}$, 1 for the answer (increasing) with explanation, since $d^2h/dt^2 = 1/200 > 0$.

Solution 4

(c) Mark scheme

- Separating variables: $\dfrac{dh}{\sqrt{h}} = -\dfrac{1}{10}\,dt$. Integrating: $\displaystyle\int \frac{dh}{\sqrt{h}} = \int -\frac{1}{10}\,dt \Rightarrow 2\sqrt{h} = -\frac{1}{10}t + C$. Using the initial condition $h(0) = 5$: $2\sqrt{5} = -\frac{1}{10}(0) + C \Rightarrow C = 2\sqrt{5}$. So $2\sqrt{h} = -\dfrac{1}{10}t + 2\sqrt{5}$, and $h(t) = \left(-\dfrac{1}{20}t + \sqrt{5}\right)^2$. Points: 1 for separating variables, 1 for antiderivatives ($2\sqrt{h}$ and $-t/10$), 1 for the constant of integration AND correctly using the initial condition $h(0) = 5$, 1 for the final expression $h(t) = (-t/20 + \sqrt{5})^2$. Note:

Solution 4

score 0/4 if no separation of variables; max 2/4 (pattern 1-1-0-0) if no constant of integration.