Past-paper walk-through

Using the First Derivative Test to Determine Relative (Local) Extrema ▪ 5.4

eXercise

Questions in this set

- 2024 Q3
- 2024 Q4
- 2023 Q4
- 2021 Q3
- 2021 Q5
- 2018 Q1
- 2016 Q3
- 2015 Q4
- 2015 Q5
- 2014 Q5

Reading the mark scheme

- **B1** a mark that stands on its own – the point is either made or not.
- **M1** a *method* mark: the right approach, even if the number is wrong.
- **A1** an *answer* mark, and it usually needs its M mark first.
- **C1** a working mark on the way to the answer.
- // separates answers the examiner accepts equally.
- **ecf** = error carried forward: a later part is marked on your own earlier answer.
- **owtte** = "or words to that effect" – the wording need not match.

Question 3

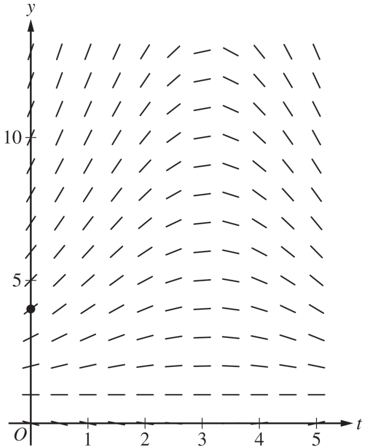
2024 ▪ Q3

The depth of seawater at a location can be modeled by the function H that satisfies the differential equation $\dfrac{dH}{dt} = \dfrac{1}{2}(H-1)\cos\left(\dfrac{t}{2}\right)$, where $H(t)$ is measured in feet and t is measured in hours after noon $(t=0)$. It is known that $H(0)=4$.

(a) A portion of the slope field for the differential equation is provided. Sketch the solution curve, $y=H(t)$, through the point $(0,4)$.

[Slope field graph; t-axis (horizontal) labeled 1 to 5, y-axis (vertical) labeled 5 and 10; a dot is plotted at $(0,4)$; short line segments show the slope at each grid point — segments are steep and negatively-sloped (like "") in the upper-right region, nearly flat in the lower region, and forward-sloped (like "/") in the upper-left region, consistent with the given differential equation.]

Question 3



Question 3

2024 ▪ Q3

The depth of seawater at a location can be modeled by the function H that satisfies the differential equation $\dfrac{dH}{dt} = \dfrac{1}{2}(H-1)\cos\left(\dfrac{t}{2}\right)$, where $H(t)$ is measured in feet and t is measured in hours after noon $(t = 0)$. It is known that $H(0) = 4$.

(b) For $0 < t < 5$, it can be shown that $H(t) > 1$. Find the value of t, for $0 < t < 5$, at which H has a critical point. Determine whether the critical point corresponds to a relative minimum, a relative maximum, or neither a relative minimum nor a relative maximum of the depth of seawater at the location. Justify your answer.

Question 3

2024 ▪ Q3

The depth of seawater at a location can be modeled by the function H that satisfies the differential equation $\dfrac{dH}{dt} = \dfrac{1}{2}(H-1)\cos\left(\dfrac{t}{2}\right)$, where $H(t)$ is measured in feet and t is measured in hours after noon $(t=0)$. It is known that $H(0)=4$.

(c) Use separation of variables to find $y = H(t)$, the particular solution to the differential equation $\dfrac{dH}{dt} = \dfrac{1}{2}(H-1)\cos\left(\dfrac{t}{2}\right)$ with initial condition $H(0)=4$.

Solution 3

(a) Mark scheme

- The solution curve must pass through $(0, 4)$ and follow the given slope field: it rises (following the steep positive-then-shallowing slopes), reaches a relative maximum near $t = \pi \approx 3.14$ (height about 9.4, matching part (b)'s critical point), then decreases for $\pi < t < 5$, staying consistent with the marked segments and extending to at least $t = 4.5$ with no obvious conflicts with the drawn slope lines. 1 point awarded only if the curve passes through $(0, 4)$, extends to at least $t = 4.5$, and has no obvious conflicts with the given slope lines (only the portion inside the given field is considered).

Solution 3

(b) Mark scheme

- Since $H(t) > 1$ on $0 < t < 5$, $\dfrac{dH}{dt} = 0$ requires $\cos\left(\dfrac{t}{2}\right) = 0$, which gives $t = \pi$. For $0 < t < \pi$, $\dfrac{dH}{dt} > 0$, and for $\pi < t < 5$, $\dfrac{dH}{dt} < 0$; therefore $t = \pi$ is the location of a relative maximum of H (the depth of seawater). 1 point for considering the sign of dH/dt (or equivalently $\cos(t/2)$), 1 point for identifying $t = \pi$ as the critical point, 1 point (only earnable with the first) for the correct classification 'relative maximum' with justification via a sign analysis of dH/dt on either side of $t = \pi$ (or an equivalent Second Derivative Test).

Solution 3

(c) Mark scheme

- Separate variables: $\dfrac{dH}{H-1} = \dfrac{1}{2}\cos\left(\dfrac{t}{2}\right)\,dt$. Integrate both sides: $\ln|H-1| = \sin\left(\dfrac{t}{2}\right) + C$. Apply $H(0) = 4$: $\ln|4-1| = \sin(0) + C \Rightarrow C = \ln 3$; since $H(0) = 4 > 1$, $|H-1| = H-1$, so $\ln(H-1) = \sin(t/2) + \ln 3$. Solve for H: $H - 1 = e^{\sin(t/2)+\ln 3} = 3e^{\sin(t/2)}$, so $H(t) = 1 + 3e^{\sin(t/2)}$. 1 point for correct separation of variables, 1 point for one correct antiderivative (either side), 1 point for the second correct antiderivative, 1 point for including the constant of integration and correctly using the initial condition $H(0) = 4$ (eligible only if the separation point and at least one antiderivative point were earned), 1 point for the

Solution 3

fully solved final answer $H(t) = 1 + 3e^{\sin(t/2)}$ (eligible only if the first four points were earned; a response with no separation of variables earns 0 of 5).

Question 4

2024 ▪ Q4

[Graph of f; x-axis labeled -6 to 7, y-axis labeled -2 to 3; the labeled point $(-6, 0.5)$ is the left endpoint of the curve; the curve rises to a horizontal tangent (peak) at $x = -2$ reaching just above $y = 2$, then descends, crossing the x-axis near $x = 4$, and continues as a straight line down to the labeled point $(6, -1)$ and beyond; the shaded region R lies in the second quadrant between the curve, the vertical line $x = -6$, and the x- and y-axes, spanning roughly $-6 \le x \le 0$.]

The graph of the differentiable function f, shown for $-6 \le x \le 7$, has a horizontal tangent at $x = -2$ and is linear for $0 \le x \le 7$. Let R be the region in the second quadrant bounded by the graph of f, the vertical line $x = -6$, and the x- and y-axes. Region R has area 12.

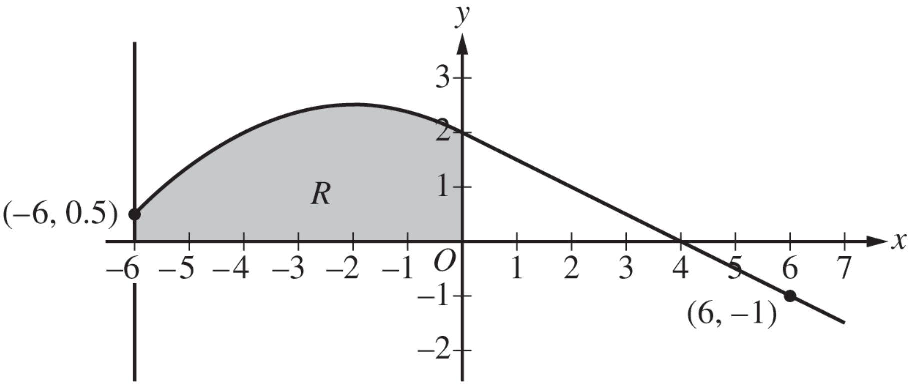
(a) The function g is defined by $g(x) = \int_0^x f(t)\,dt$. Find the values of $g(-6)$, $g(4)$, and $g(6)$.

Question 4

(b) For the function g defined in part (a), find all values of x in the interval $0 \leq x \leq 6$ at which the graph of g has a critical point. Give a reason for your answer.

(c) The function h is defined by $h(x) = \int_{-6}^{x} f'(t)\, dt$. Find the values of $h(6)$, $h'(6)$, and $h''(6)$. Show the work that leads to your answers.

Question 4



Graph of f

Solution 4

(a) Mark scheme

- Using the graph of f and $g(x) = \int_0^x f(t)\,dt$:
  $g(-6) = \int_0^{-6} f(t)\,dt = -\int_{-6}^{0} f(t)\,dt = -12$ (since region R, the area under f from -6 to 0, has area 12, given in the problem). $g(4) = \int_0^4 f(t)\,dt = \frac{1}{2}(4)(2) = 4$ (triangle area under the linear piece of f from $x = 0$, where $f(0) = 2$, down to where it crosses the x-axis at $x = 4$).
  $g(6) = \int_0^6 f(t)\,dt = \frac{1}{2}(4)(2) - \frac{1}{2}(2)(1) = 4 - 1 = 3$ (adding the negative triangle from $x = 4$ to $x = 6$, where $f(6) = -1$). 1 point each for $g(-6) = -12$, $g(4) = 4$, $g(6) = 3$ (supporting work not required, but if shown it must be correct to earn the point; unlabeled values are read in the order $g(-6), g(4), g(6)$).

Solution 4

Solution 4

(b)

Mark scheme

- By the Fundamental Theorem of Calculus, $g'(x) = f(x)$. Setting $g'(x) = f(x) = 0$ on $0 \leq x \leq 6$ gives $x = 4$ (where the graph of f crosses the x-axis, matching part (a)). Therefore the graph of g has a critical point at $x = 4$. 1 point for explicitly making the connection $g' = f$, 1 point for the answer $x = 4$ with a valid reason (e.g. $f(4) = 0$ and f changes sign there).

Solution 4

(c) Mark scheme

- $h(x) = \int_{-6}^{x} f'(t)\,dt$, so by the Fundamental Theorem of Calculus $h(6) = \int_{-6}^{6} f'(t)\,dt = f(6) - f(-6) = -1 - 0.5 = -1.5$. $h'(x) = f'(x)$, so $h'(6) = f'(6) = -\dfrac{1}{2}$ (the slope of the linear piece of f on $0 \leq x \leq 7$, which passes through $(0, 2)$ and $(6, -1)$). $h''(x) = f''(x)$, so $h''(6) = f''(6) = 0$ (since f is linear on $[0, 7]$, its second derivative is 0 there). 1 point for using the Fundamental Theorem of Calculus to evaluate $h(6)$ as $f(6) - f(-6)$, 1 point for the value $h(6) = -1.5$ with supporting work, 1 point for $h'(6) = -1/2$, 1 point for $h''(6) = 0$.

Question 4

2023 ▪ Q4

The function f is defined on the closed interval $[-2, 8]$ and satisfies $f(2) = 1$. The graph of f', the derivative of f, consists of two line segments and a semicircle, as shown in the figure.
[Graph of f': x-axis from -2 to 8 with gridlines at every integer, y-axis from -2 to 2 with gridlines at every integer. The graph consists of: a line segment from $(-2, 2)$ down to $(0, -2)$; a line segment from $(0, -2)$ up to $(4, 2)$; and a semicircle (bulging downward, below the x-axis) connecting $(4, 2)$ down through the x-axis and back up to $(8, 2)$, dipping to a minimum of about 0 at $x = 6$. Solid dots are marked at $(-2, 2)$, $(0, -2)$, $(4, 2)$, and $(8, 2)$.]
(a) Does f have a relative minimum, a relative maximum, or neither at $x = 6$? Give a reason for your answer.

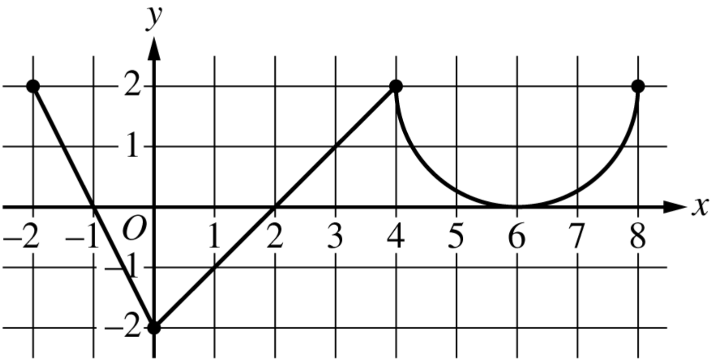
Question 4

(b) On what open intervals, if any, is the graph of f concave down? Give a reason for your answer.

(c) Find the value of $\lim_{x \to 2} \dfrac{6f(x) - 3x}{x^2 - 5x + 6}$, or show that it does not exist. Justify your answer.

(d) Find the absolute minimum value of f on the closed interval $[-2, 8]$. Justify your answer.

Question 4



Graph of f'

Solution 4

(a)

Mark scheme

- $f'(x) > 0$ on (2,6) and $f'(x) > 0$ on (6,8), so f'(x) does not change sign at x=6; therefore f has neither a relative maximum nor a relative minimum at x=6 (1 pt: correct answer — 'neither' — with the reason that f'(x) does not change sign at x=6).

Solution 4

(b)

Mark scheme

- The graph of f is concave down on (-2,0) and (4,6) because f' is decreasing on these intervals (1 pt: correct intervals (-2,0) and (4,6) — with or without endpoints; 1 pt: reason, must correctly discuss the behavior/slopes of f').

Solution 4

(c) Mark scheme

- Because f is differentiable, hence continuous, at x=2, $\lim_{x\to 2} f(x) = f(2) = 1$; so $\lim_{x\to 2}(6f(x) - 3x) = 6(1) - 3(2) = 0$ and $\lim_{x\to 2}(x^2 - 5x + 6) = 0$ (1 pt: both limits of numerator and denominator shown to equal 0; a limit presented as literally equal to 0/0 does not earn this point). Since the limit is of indeterminate form 0/0, L'Hospital's Rule applies (1 pt: uses L'Hospital's Rule — at least one correct derivative shown in the limit of a ratio of derivatives). $\lim_{x\to 2} \dfrac{6f(x) - 3x}{x^2 - 5x + 6} = \lim_{x\to 2} \dfrac{6f'(x) - 3}{2x - 5} = \dfrac{6(0) - 3}{2(2) - 5} = \dfrac{-3}{-1} = 3$ (1 pt: correct answer 3 with supporting work).

Solution 4

(d)

Mark scheme

- $f'(x) = 0 \Rightarrow x = -1, 2, 6$ (1 pt: considers f'(x)=0 — merely listing the zeros of f' without stating f'=0 is not sufficient). Since f is continuous on [-2,8], the candidates for the location of an absolute minimum are x=-2,-1,2,6,8, with $f(-2) = 3$, $f(-1) = 4$, $f(2) = 1$, $f(6) = 7 - \pi$, $f(8) = 11 - 2\pi$ (1 pt: justification — evaluate f at all critical points and both endpoints, or validly eliminate some by argument). The absolute minimum value of f is $f(2) = 1$ (1 pt: answer — must state the minimum VALUE 1, not merely its location x=2).

Question 3

2021 ▪ Q3

[Graph showing a curve in the first quadrant starting at the origin O, rising to a rounded peak, then curving back down to meet the x-axis; x-axis and y-axis labeled, no numeric scale marks shown.]

A company designs spinning toys using the family of functions $y = cx\sqrt{4 - x^2}$, where c is a positive constant. The figure above shows the region in the first quadrant bounded by the x-axis and the graph of $y = cx\sqrt{4 - x^2}$, for some c. Each spinning toy is in the shape of the solid generated when such a region is revolved about the x-axis. Both x and y are measured in inches.

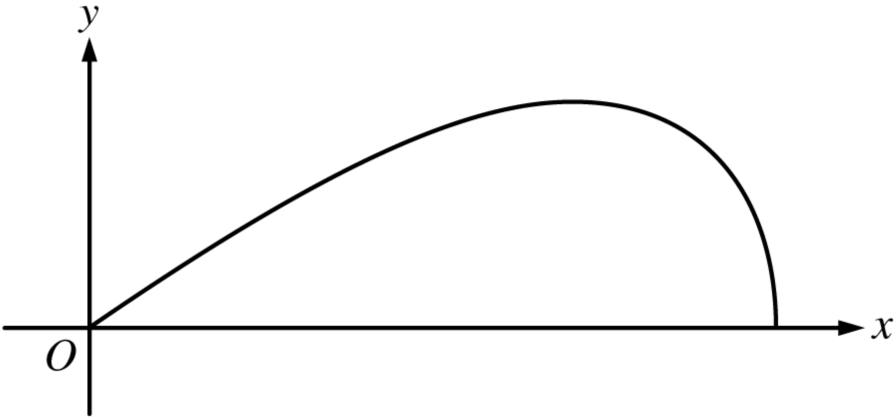
(a) Find the area of the region in the first quadrant bounded by the x-axis and the graph of $y = cx\sqrt{4 - x^2}$ for $c = 6$.

Question 3

(b) It is known that, for $y = cx\sqrt{4-x^2}$, $\dfrac{dy}{dx} = \dfrac{c(4-2x^2)}{\sqrt{4-x^2}}$. For a particular spinning toy, the radius of the largest cross-sectional circular slice is 1.2 inches. What is the value of c for this spinning toy?

(c) For another spinning toy, the volume is 2π cubic inches. What is the value of c for this spinning toy?

Question 3



Solution 3

(a) Mark scheme

- For $c = 6$: $6x\sqrt{4-x^2} = 0 \Rightarrow x = 0, 2$, so Area $= \displaystyle\int_0^2 6x\sqrt{4-x^2}\,dx$. Using $u = 4 - x^2$: $\int_0^2 6x\sqrt{4-x^2}\,dx = 3\int_0^4 u^{1/2}\,du = 2u^{3/2}\Big|_0^4 = 2\cdot 8 = 16$. The area of the region is 16 square inches. Point 1: presenting $cx\sqrt{4-x^2}$ or $6x\sqrt{4-x^2}$ as the integrand of a definite integral (limits need not be correct). Point 2: a correct antiderivative of the form $Ax\sqrt{4-x^2}\ldots$ [presented as $Au^{3/2}$-type expression] for any nonzero constant (independent of point 1). Point 3: the final answer 16 (or equivalent), only earned if point 2 was earned, using the correct limits.

Solution 3

(b) Mark scheme

- The largest cross-sectional radius occurs where $\dfrac{dy}{dx} = 0$:
  $\dfrac{dy}{dx} = \dfrac{c(4-2x^2)}{\sqrt{4-x^2}} = 0 \Rightarrow x = \sqrt{2}$. Then $y = c\sqrt{2}\sqrt{4-(\sqrt{2})^2} = 2c$. Setting $2c = 1.2$ gives $c = 0.6$. Point 1: setting $\frac{dy}{dx} = 0$ (or the equivalent $c(4-2x^2) = 0$); an unsupported $x = \sqrt{2}$ alone does not earn it. Point 2: the final answer $c = 0.6$ with supporting work (can be earned without point 1).

Solution 3

(c) Mark scheme

- Volume $= \int_0^2 \pi \left(cx\sqrt{4-x^2}\right)^2 dx = \pi c^2 \int_0^2 x^2(4-x^2)\, dx =$ $\pi c^2 \int_0^2 (4x^2 - x^4)\, dx = \pi c^2 \left(\frac{4}{3}x^3 - \frac{1}{5}x^5\right)\Big|_0^2 = \pi c^2 \left(\frac{32}{3} - \frac{32}{5}\right) = \frac{64\pi c^2}{15}$. Setting $\frac{64\pi c^2}{15} = 2\pi$ gives $c^2 = \frac{15}{32}$, so $c = \sqrt{15/32}$. Point 1: integrand of the form $A(x\sqrt{4-x^2})^2$ in a definite integral, any limits, any nonzero constant A (mishandling c makes it ineligible for point 4). Point 2: correct limits $x = 0, 2$ and constant π (not 2π) as part of an integral — earnable without point 1. Point 3: correct antiderivative of the presented integrand, form $A(x\sqrt{4-x^2})^2$-derived

Solution 3

polynomial, for any nonzero A — needed for point 4. Point 4: final answer $c = \sqrt{15/32}$ (need not be simplified); cannot be earned without point 3.

Question 5

2021 ▪ Q5

Consider the function $y = f(x)$ whose curve is given by the equation $2y^2 - 6 = y \sin x$ for $y > 0$.

(a) Show that $\dfrac{dy}{dx} = \dfrac{y \cos x}{4y - \sin x}$.

(b) Write an equation for the line tangent to the curve at the point $(0, \sqrt{3})$.

(c) For $0 \leq x \leq \pi$ and $y > 0$, find the coordinates of the point where the line tangent to the curve is horizontal.

(d) Determine whether f has a relative minimum, a relative maximum, or neither at the point found in part (c). Justify your answer.

Solution 5

(a) Mark scheme

- Implicitly differentiate $2y^2 - 6 = y\sin x$: $\dfrac{d}{dx}(2y^2 - 6) = \dfrac{d}{dx}(y\sin x) \Rightarrow 4y\dfrac{dy}{dx} = \dfrac{dy}{dx}\sin x + y\cos x$. Then $4y\dfrac{dy}{dx} - \dfrac{dy}{dx}\sin x = y\cos x \Rightarrow \dfrac{dy}{dx}(4y - \sin x) = y\cos x \Rightarrow \dfrac{dy}{dx} = \dfrac{y\cos x}{4y - \sin x}$, as required. Point 1: correct implicit differentiation of $2y^2 - 6 = y\sin x$ (alternative notations for dy/dx such as y' are fine) — required for point 2. Point 2: verification, i.e. reaching $\frac{dy}{dx}(4y - \sin x) = y\cos x$ is sufficient with no subsequent errors.

Solution 5

(b)

Mark scheme

- At the point $(0, \sqrt{3})$: $\dfrac{dy}{dx} = \dfrac{\sqrt{3}\cos 0}{4\sqrt{3} - \sin 0} = \dfrac{1}{4}$. An equation for the tangent line is $y = \sqrt{3} + \dfrac{1}{4}x$. Point 1: any correct tangent line equation; no supporting work is required and the slope value need not be simplified.

Solution 5

(c) Mark scheme

- $\dfrac{dy}{dx} = \dfrac{y\cos x}{4y - \sin x} = 0 \Rightarrow y\cos x = 0$ and $4y - \sin x \neq 0$. Since $y\cos x = 0$ and $y > 0 \Rightarrow \cos x = 0 \Rightarrow x = \dfrac{\pi}{2}$ (on $0 \leq x \leq \pi$). When $x = \pi/2$: $y\sin(\pi/2) = 2y^2 - 6 \Rightarrow y = 2y^2 - 6 \Rightarrow 2y^2 - y - 6 = 0 \Rightarrow (2y+3)(y-2) = 0 \Rightarrow y = 2$ (since $y > 0$). Check: $4y - \sin x = 8 - 1 = 7 \neq 0$. The tangent line is horizontal at $\left(\dfrac{\pi}{2}, 2\right)$. Point 1: sets $\frac{dy}{dx} = 0$ (or equivalently $y\cos x = 0$ or $\cos x = 0$). Point 2: commits to $x = \pi/2$ only (any extraneous additional x-values forfeit this point; y-values presented are not considered here). Point 3: $y = 2$ (not earned without point 2; coordinates need not be presented as an ordered pair).

Solution 5

(d) Mark scheme

- $\dfrac{d^2y}{dx^2} = \dfrac{(4y - \sin x)\left(\frac{dy}{dx}\cos x - y\sin x\right) - (y\cos x)\left(4\frac{dy}{dx} - \cos x\right)}{(4y - \sin x)^2}$. At $x = \pi/2, y = 2$ (with $dy/dx = 0$): $\dfrac{d^2y}{dx^2} = \dfrac{(7)(-2) - (0)(0)}{49} = \dfrac{-2}{7} < 0$. Since $\dfrac{dy}{dx} = 0$ and $\dfrac{d^2y}{dx^2} < 0$ at $\left(\frac{\pi}{2}, 2\right)$, f has a RELATIVE MAXIMUM there (Second Derivative Test). [Alternate solution via First Derivative Test: near $(\pi/2, 2)$, $4y - \sin x > 0$ and $y > 0$, so $\frac{dy}{dx} = \frac{y\cos x}{4y - \sin x}$ changes from positive to negative at $x = \pi/2$, giving a relative maximum by the First Derivative Test.] Point 1: attempt to use the quotient (or product) rule to find d^2y/dx^2 [or: considers the

Solution 5

sign of $4y - \sin x$]. Point 2: correctly finds d^2y/dx^2 and evaluates it as negative at the point [or: shows dy/dx changes sign $+$ to $-$ at $x = \pi/2$ — can be earned without point 1]. Point 3: correct conclusion (relative maximum) with justification, consistent with a reported nonzero value/sign obtained using $dy/dx = 0$; a response concluding "minimum" earns no point-3 credit.

Question 1

2018 ▪ Q1

People enter a line for an escalator at a rate modeled by the function r given by

$$r(t) = \begin{cases} 44\left(\frac{t}{100}\right)^3\left(1-\frac{t}{300}\right)^7 & \text{for } 0 \leq t \leq 300 \\ 0 & \text{for } t > 300, \end{cases}$$

where $r(t)$ is measured in people per second and t is measured in seconds. As people get on the escalator, they exit the line at a constant rate of 0.7 person per second. There are 20 people in line at time $t = 0$.

(a) How many people enter the line for the escalator during the time interval $0 \leq t \leq 300$?

(b) During the time interval $0 \leq t \leq 300$, there are always people in line for the escalator. How many people are in line at time $t = 300$?

Question 1

(c) For $t > 300$, what is the first time t that there are no people in line for the escalator?

(d) For $0 \leq t \leq 300$, at what time t is the number of people in line a minimum? To the nearest whole number, find the number of people in line at this time. Justify your answer.

Solution 1

(a)

Mark scheme

- $\int_0^{300} r(t)\, dt = 270$. According to the model, 270 people enter the line for the escalator during $0 \leq t \leq 300$. (1 pt: setting up the integral; 1 pt: the answer 270.)

Solution 1

(b)

Mark scheme

- $20 + \int_0^{300} (r(t) - 0.7)\, dt = 20 + \int_0^{300} r(t)\, dt - 0.7 \cdot 300 = 80$. According to the model, 80 people are in line at time $t = 300$. (1 pt: considers the rate out, i.e. subtracts $0.7t$; 1 pt: the answer 80.)

Solution 1

(c)

Mark scheme

- Based on part (b), the number of people in line at $t = 300$ is 80. The first time $t > 300$ with no people in line is $300 + \dfrac{80}{0.7} = 414.286$ (or 414.285) seconds. (1 pt: the answer.)

Solution 1

(d)

Mark scheme

- The number in line for $0 \leq t \leq 300$ is $20 + \int_0^t r(x)\,dx - 0.7t$. Setting $r(t) - 0.7 = 0$ gives candidates $t_1 = 33.013298$ and $t_2 = 166.574719$ (only t_1 is in $[0, 300]$ as an interior candidate). Evaluating the count at $t = 0$ (20), $t = t_1 = 33.013$ (3.803), $t = t_2$ (158.070, outside relevant minimum), and $t = 300$ (80) shows the minimum is at $t = 33.013$ seconds, with 4 people in line (nearest whole number). (1 pt: considers $r(t) - 0.7 = 0$; 1 pt: identifies $t = 33.013$; 1 pt: answer of 4 people; 1 pt: justification via a candidates test.)

Question 3

2016 ▪ Q3

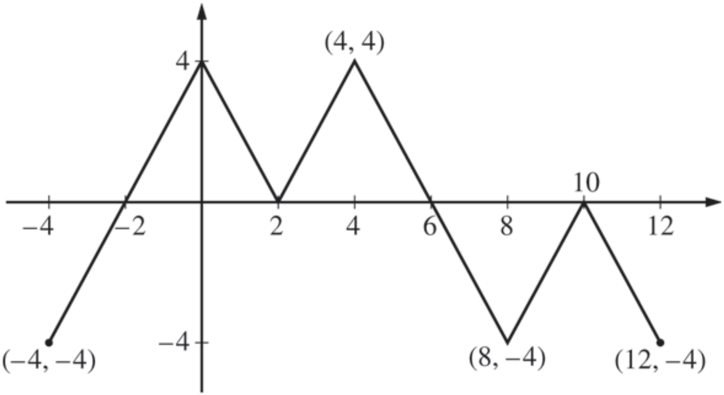
[Graph of the piecewise-linear function f, labeled "Graph of f"; x-axis from -4 to 12 with gridlines/labels at $-4, -2, 2, 4, 6, 8, 10, 12$; y-axis showing 4 and -4. The graph consists of straight line segments connecting the labeled points: $(-4, -4)$ up to $(0, 4)$, down to $(2, 0)$, up to $(4, 4)$, down to $(8, -4)$, up to $(10, 0)$, down to $(12, -4)$.]
The figure above shows the graph of the piecewise-linear function f. For $-4 \leq x \leq 12$, the function g is defined by $g(x) = \int_2^x f(t)\,dt$.

(a) Does g have a relative minimum, a relative maximum, or neither at $x = 10$? Justify your answer.
(b) Does the graph of g have a point of inflection at $x = 4$? Justify your answer.
(c) Find the absolute minimum value and the absolute maximum value of g on the interval $-4 \leq x \leq 12$. Justify your answers.

Question 3

(d) For $-4 \leq x \leq 12$, find all intervals for which $g(x) \leq 0$.

Question 3



Graph of f

Solution 3

(a)

Mark scheme

- Since g(x)=$\int_2^x f(t)\,dt, g'(x) = f(x)[thisrecognitionpointmaybeearnedinpart(a),(b),(c),or(d)-10, sinceg'(x) = f(x)andf(x) \leq 0(doesnotchangesign)for8 \leq x \leq 12.Points : 1forrecognizingg'(x) = f(x)(awardhereifnotalreadycreditedinanotherpart), 1forthecorrectanswer(neither)withvalidjus$

Solution 3

(b)

Mark scheme

- The graph of g has a point of inflection at x=4, since g'(x)=f(x) is increasing for 2$\leq x \leq 4 anddecreasingfor 4 \leq x \leq 8 (sog' changesfromincreasingtodecreasingatx = 4, meaninggchangesconcavitythere). Points : 1 forthecorrectanswer(yes)withvalidjustification.$

Solution 3

(c)

Mark scheme

- g'(x)=f(x) changes sign only at x=-2 and x=6. Evaluating g at the candidates and endpoints: g(-4)=-4, g(-2)=-8, g(6)=8, g(12)=-4. On -4$\leq$ x $\leq$ $12, theabsoluteminimumvalueisg(-2) = -8andtheabsolutemaximumvalueisg(6) = 8.Points : 1foridentifyingx = -2andx = 6ascandidates(whereg'changessign), 1foralsoconsideringtheendpointsx = -4andx = 12, 2forthecorrectabsoluteminimumandmaximumvalueswithjustification.$

Solution 3

(d)

Mark scheme

- g(x)$\leq 0 for - 4 \leq x \leq 2 and 10 \leq x \leq 12, readfromthevalues/signofgestablishedinpart(c)togetherwiththegraphoff. Points : 2forthecorrectpairofintervals.$

Question 4

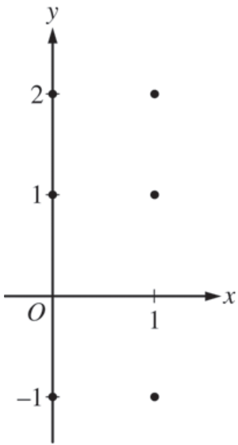
2015 ▪ Q4

Consider the differential equation $\dfrac{dy}{dx} = 2x - y$.

(a) On the axes provided, sketch a slope field for the given differential equation at the six points indicated.

[Slope-field axes labeled y (vertical) and x (horizontal), origin O. Six points are marked with dots, at coordinates $(1, 2)$, $(1, 1)$, $(1, -1)$, and their positions relative to axis tick marks at $y = 2$, $y = 1$, $y = -1$ on the y-axis and $x = 1$ on the x-axis. (The six indicated points are at $x = 1$ paired with $y = 2, 1, -1$, and at $x = 2$ paired with $y = 2, 1, -1$, based on the two columns of dots shown above $y = 2$, $y = 1$, and $y = -1$.)]

Question 4



Question 4

2015 ▪ Q4

Consider the differential equation $\frac{dy}{dx} = 2x - y$.

(b) Find $\frac{d^2y}{dx^2}$ in terms of x and y. Determine the concavity of all solution curves for the given differential equation in Quadrant II. Give a reason for your answer.

Question 4

2015 ▪ Q4

Consider the differential equation $\frac{dy}{dx} = 2x - y$.

(c) Let $y = f(x)$ be the particular solution to the differential equation with the initial condition $f(2) = 3$. Does f have a relative minimum, a relative maximum, or neither at $x = 2$? Justify your answer.

Question 4

2015 ▪ Q4

Consider the differential equation $\frac{dy}{dx} = 2x - y$.

(d) Find the values of the constants m and b for which $y = mx + b$ is a solution to the differential equation.

Solution 4

(a)

Mark scheme

- Slope field at the six indicated points using $dy/dx = 2x - y$. At $x = 0$: (0,2) slope $= -2$, (0,1) slope $= -1$, (0,−1) slope $= 1$. At $x = 1$: (1,2) slope $= 0$, (1,1) slope $= 1$, (1,−1) slope $= 3$. Points: 1 for correct slope segments drawn at the three points where $x = 0$, 1 for correct slope segments drawn at the three points where $x = 1$. (This is a drawing task — grade by checking the drawn segment directions/steepness at each of the six given points match these computed slopes, not a numeric or algebraic answer.)

Solution 4

(b)

Mark scheme

- $d^2y/dx^2 = 2 - dy/dx = 2 - (2x - y) = 2 - 2x + y$. In Quadrant II, $x < 0$ and $y > 0$, so $2 - 2x + y > 0$ (each term is positive). Therefore all solution curves are concave up in Quadrant II. Points: 1 for the correct expression $d^2y/dx^2 = 2 - 2x + y$, 1 for the correct conclusion (concave up) with a valid reason (sign argument in Quadrant II).

Solution 4

(c)

Mark scheme

- dy/dx at $(x, y) = (2, 3)$ is $2(2) - 3 = 1 \neq 0$. Therefore f has neither a relative minimum nor a relative maximum at $x = 2$, since the derivative is nonzero there. Points: 1 for considering dy/dx at $(2, 3)$, 1 for the correct conclusion (neither min nor max) with justification ($dy/dx \neq 0$ at that point).

Solution 4

(d)

Mark scheme

- If $y = mx + b$, then $dy/dx = d/dx(mx + b) = m$. Substituting into $2x - y = m$ gives $2x - (mx + b) = m$, i.e. $(2 - m)x - (m + b) = 0$ for all x, so $2 - m = 0 \quad m = 2$, and $-(m + b) = 0 \quad b = -m = -2$. Therefore $m = 2$ and $b = -2$. Points: 1 for $d/dx(mx + b) = m$, 1 for the equation $2x - y = m$ (substituting $y = mx + b$ into the differential equation, or equivalent), 1 for the correct answer $m = 2$, $b = -2$.

Question 5

2015 ▪ Q5

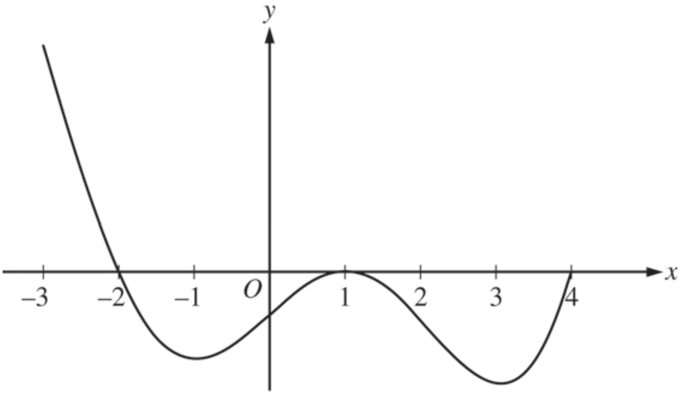
[Graph of f': axes labeled y (vertical) and x (horizontal), with tick marks on the x-axis at $-3, -2, -1, O, 1, 2, 3, 4$. The curve begins high on the left (above $x = -3$), decreases steeply, crosses the x-axis just before $x = -1$, continues down to a local minimum between $x = -1$ and O, rises back up crossing the x-axis near $x = 1$, reaches a local maximum between $x = 1$ and $x = 2$, then decreases again crossing the x-axis before $x = 3$, reaches a local minimum between $x = 3$ and $x = 4$, then rises steeply up to $x = 4$. Labeled "Graph of f'".]

The figure above shows the graph of f', the derivative of a twice-differentiable function f, on the interval $[-3, 4]$. The graph of f' has horizontal tangents at $x = -1$, $x = 1$, and $x = 3$. The areas of the regions bounded by the x-axis and the graph of f' on the intervals $[-2, 1]$ and $[1, 4]$ are 9 and 12, respectively.

Question 5

(a) Find all x-coordinates at which f has a relative maximum. Give a reason for your answer.
(b) On what open intervals contained in $-3 < x < 4$ is the graph of f both concave down and decreasing? Give a reason for your answer.
(c) Find the x-coordinates of all points of inflection for the graph of f. Give a reason for your answer.
(d) Given that $f(1) = 3$, write an expression for $f(x)$ that involves an integral. Find $f(4)$ and $f(-2)$.

Question 5



Graph of f'

Solution 5

(a)

Mark scheme

- $f'(x) = 0$ at $x = -2$, $x = 1$, and $x = 4$. $f'(x)$ changes from positive to negative only at $x = -2$. Therefore f has a relative maximum at $x = -2$ (this is the only x-coordinate at which f has a relative maximum). Points: 1 for identifying $x = -2$, 1 for the answer with correct reason (sign change of f' from + to −).

Solution 5

(b)

Mark scheme

- The graph of f is concave down and decreasing on the intervals $-2 < x < -1$ and $1 < x < 3$, because f' is decreasing and negative on these intervals. Points: 1 for identifying the correct intervals $(-2, -1)$ and $(1, 3)$, 1 for the correct reason (f' decreasing gives concave down; f' negative gives decreasing).

Solution 5

(c)

Mark scheme

- The graph of f has points of inflection at $x = -1$ and $x = 3$, because f' changes from decreasing to increasing at these points. The graph of f also has a point of inflection at $x = 1$, because f' changes from increasing to decreasing at this point. So the inflection x-coordinates are -1, 1, and 3. Points: 1 for identifying $x = -1$, 1, and 3, 1 for the correct reason (f' changes from increasing to decreasing, or decreasing to increasing, at each point).

Solution 5

(d)

Mark scheme

- f(x) = 3 + $\int$ ^x f'(t) dt (using f(1) = 3). f(4) = 3 + $\int$ ^4 f'(t) dt = 3 + (−12) = −9 (the integral is the negative of the area 12 on [1, 4], since f' is negative there). f(−2) = 3 + $\int$ ^(−2) f'(t) dt = 3 − $\int$_(−2)^1 f'(t) dt = 3 − (−9) = 12 (the integral over [−2, 1] is the negative of the area 9, since f' is negative there). Points: 1 for the correct integrand f'(t) in an expression for f, 1 for a correct expression for f(x) (f(x) = 3 + $\int$ ^x f'(t) dt), 1 for both correct final values f(4) = −9 and f(−2) = 12.

Question 5

2014 ▪ Q5

The twice-differentiable functions f and g are defined for all real numbers x. Values of f, f', g, and g' for various values of x are given in the table above.

x	-2	$-2 < x < -1$	-1	$-1 < x < 1$	1	$1 < x < 3$	3
$f(x)$	12	Positive	8	Positive	2	Positive	7
$f'(x)$	-5	Negative	0	Negative	0	Positive	$\frac{1}{2}$
$g(x)$	-1	Negative	0	Positive	3	Positive	1
$g'(x)$	2	Positive	$\frac{3}{2}$	Positive	0	Negative	-2

Question 5

(a) Find the x-coordinate of each relative minimum of f on the interval $[-2, 3]$. Justify your answers.
(b) Explain why there must be a value c, for $-1 < c < 1$, such that $f''(c) = 0$.
(c) The function h is defined by $h(x) = \ln(f(x))$. Find $h'(3)$. Show the computations that lead to your answer.
(d) Evaluate $\int_{-2}^{3} f'(g(x))g'(x)\,dx$.

Question 5

x	-2	$-2 < x < -1$	-1	$-1 < x < 1$	1	$1 < x < 3$	3
$f(x)$	12	Positive	8	Positive	2	Positive	7
$f'(x)$	-5	Negative	0	Negative	0	Positive	$\frac{1}{2}$
$g(x)$	-1	Negative	0	Positive	3	Positive	1
$g'(x)$	2	Positive	$\frac{3}{2}$	Positive	0	Negative	-2

Solution 5

(a)

Mark scheme

- $x = 1$ is the only critical point at which f' changes sign from negative to positive (f' is negative on $-1<x<1$ and positive on $1<x<3$, so f' changes sign at $x=1$). Therefore, f has a relative minimum at $x = 1$. Points: 1 for the correct x-value ($x=1$) together with a valid justification (first-derivative sign change from negative to positive at $x=1$).

Solution 5

(b)

Mark scheme

- f' is differentiable, so f' is continuous on [-1,1]. f'(1)-f'(-1) $\frac{}{1-(-1)=\frac{0-0}{2}=0. \mathit{Therefore, by the Mean Value Theorem applied to f' on} [-1,1], \mathit{there is at least one value} c, -1<c<1, \mathit{suchtha}}$

Solution 5

(c) Mark scheme

- h(x) = $\ln(f(x)).\mathit{Bythechainrule}, h'(x) = \frac{1}{f(x)} \cdot f'(x).h'(3) = \frac{1}{f(3)} \cdot f'(3) = \frac{1}{7} \cdot \frac{1}{2} = \frac{1}{14}.\mathit{Points}: 2\mathit{foracorrectexpressionforh}'(x)(\mathit{chainruleforln}(f(x))), 1\mathit{forthecorrectfinalanswerh}'(3) = 1/14.$

Solution 5

(d)

Mark scheme

- $\int_{-2}^{3} f'(g(x))g'(x)\,dx = \Big[f(g(x))\Big]_{x=-2}^{x=3} = f(g(3)) - f(g(-2)) = f(1) - f(-1) = 2 - 8 = -6, usingtheFundamentalTheoremofCalculus(antiderivativef(g(x))viathechainrule/u-substitutionu = g(x))withtablevaluesg(3) = 1, g(-2) = -1, f(1) = 2, f(-1) = 8. Points : 2forcorrectlyapplyingtheFundamentalTheoremofCalculus(recognizingf(g(x))astheantiderivat$
  $6.$