Past-paper walk-through

Approximating Values of a Function Using Local Linearity and Linearization ▪ 4.6

eXercise

Questions in this set

- 2026 Q3
- 2025 Q6
- 2024 Q5
- 2023 Q3
- 2022 Q5
- 2018 Q6
- 2017 Q4
- 2016 Q4
- 2014 Q1
- 2014 Q6

Reading the mark scheme

- **B1** a mark that stands on its own – the point is either made or not.
- **M1** a *method* mark: the right approach, even if the number is wrong.
- **A1** an *answer* mark, and it usually needs its M mark first.
- **C1** a working mark on the way to the answer.
- // separates answers the examiner accepts equally.
- **ecf** = error carried forward: a later part is marked on your own earlier answer.
- **owtte** = "or words to that effect" – the wording need not match.

Question 3

2026 ▪ Q3

A pie is taken from a hot oven and put on a table. The internal temperature of the pie at time t minutes can be modeled by the function H that satisfies the differential equation $\frac{dH}{dt} = -\frac{1}{15}(H - 20)$, where $H(t)$ is measured in degrees Celsius and $H(0) = 75$. For $t > 0$, it is known that $20 < H(t) < 75$.

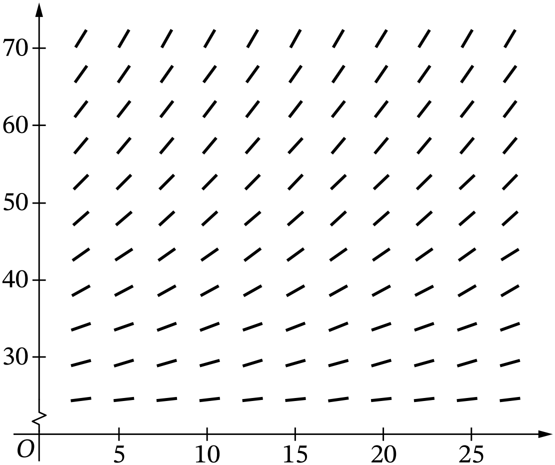
(a) Explain why the following could not be a slope field for the differential equation $\frac{dH}{dt} = -\frac{1}{15}(H - 20)$.

[Slope field diagram: H-axis (vertical) from 30 to 70+ in increments of 10, t-axis (horizontal) from 0 to 25+ in increments of 5. At every grid point across the full range of t shown, the line segments have the same positive slope for a given horizontal row and the slopes appear to depend only on t (segments in each row are parallel and tilt

Question 3

more steeply positive at greater height H, but they do not appear to flatten toward zero as $H \to 20$ nor do the segments vary with t within a row as required) — segments are drawn as short parallel dashes tilting up-and-to-the-right throughout, uniformly across each horizontal band of H values, without change as t increases.]

Question 3



Question 3

2026 ▪ Q3

A pie is taken from a hot oven and put on a table. The internal temperature of the pie at time t minutes can be modeled by the function H that satisfies the differential equation $\frac{dH}{dt} = -\frac{1}{15}(H - 20)$, where $H(t)$ is measured in degrees Celsius and $H(0) = 75$. For $t > 0$, it is known that $20 < H(t) < 75$.

(b) Find the slope of the line tangent to the graph of H at time $t = 0$. Show the work that leads to your answer.

Question 3

2026 ▪ Q3

A pie is taken from a hot oven and put on a table. The internal temperature of the pie at time t minutes can be modeled by the function H that satisfies the differential equation $\frac{dH}{dt} = -\frac{1}{15}(H - 20)$, where $H(t)$ is measured in degrees Celsius and $H(0) = 75$. For $t > 0$, it is known that $20 < H(t) < 75$.

(c) It can be shown that $\frac{d^2H}{dt^2} = \frac{1}{225}(H - 20)$. The line tangent to the graph of H at time $t = 0$ is used to approximate $H(5)$, the internal temperature of the pie at time $t = 5$. Is this approximation an overestimate or an underestimate for the actual value of $H(5)$? Give a reason for your answer.

Question 3

2026 ▪ Q3

A pie is taken from a hot oven and put on a table. The internal temperature of the pie at time t minutes can be modeled by the function H that satisfies the differential equation $\frac{dH}{dt} = -\frac{1}{15}(H - 20)$, where $H(t)$ is measured in degrees Celsius and $H(0) = 75$. For $t > 0$, it is known that $20 < H(t) < 75$.

(d) Use separation of variables to find an expression for $H(t)$, the particular solution to the given differential equation with initial condition $H(0) = 75$.

Question 6

2025 ▪ Q6

Consider the curve G defined by the equation $y^3 - y^2 - y + \frac{1}{4}x^2 = 0$.

(a) Show that $\dfrac{dy}{dx} = \dfrac{-x}{2(3y^2 - 2y - 1)}$.

Question 6

2025 ▪ Q6

Consider the curve G defined by the equation $y^3 - y^2 - y + \frac{1}{4}x^2 = 0$.

(b) There is a point P on the curve G near $(2, -1)$ with x-coordinate 1.6. Use the line tangent to the curve at $(2, -1)$ to approximate the y-coordinate of point P.

Question 6

2025 ▪ Q6

Consider the curve G defined by the equation $y^3 - y^2 - y + \frac{1}{4}x^2 = 0$.

(c) For $x > 0$ and $y > 0$, there is a point S on the curve G at which the line tangent to the curve at that point is vertical. Find the y-coordinate of point S. Show the work that leads to your answer.

Question 6

2025 ▪ Q6

Consider the curve G defined by the equation $y^3 - y^2 - y + \frac{1}{4}x^2 = 0$.

(d) A particle moves along the curve H defined by the equation $2xy + \ln y = 8$. At the instant when the particle is at the point $(4, 1)$, $\dfrac{dx}{dt} = 3$. Find $\dfrac{dy}{dt}$ at that instant. Show the work that leads to your answer.

Solution 6

(a) Mark scheme

- Differentiate $y^3 - y^2 - y + \frac{1}{4}x^2 = 0$ implicitly with respect to x:
  $3y^2\frac{dy}{dx} - 2y\frac{dy}{dx} - \frac{dy}{dx} + \frac{x}{2} = 0 \Rightarrow \left(3y^2 - 2y - 1\right)\frac{dy}{dx} = -\frac{x}{2} \Rightarrow \frac{dy}{dx} = \frac{-x}{2(3y^2 - 2y - 1)}$.
  Point 1: correct implicit differentiation of the given equation (may use y' in place of $\frac{dy}{dx}$). Point 2: correctly verifies/derives the given result $\frac{dy}{dx} = \frac{-x}{2(3y^2 - 2y - 1)}$ (it suffices to present $(3y^2 - 2y - 1)\frac{dy}{dx} = -\frac{x}{2}$ with no subsequent errors); requires Point 1.

Solution 6

(b) Mark scheme

- Slope of the tangent line at $(2,-1)$: $\left.\frac{dy}{dx}\right|_{(2,-1)} = \frac{-2}{2(3(-1)^2-2(-1)-1)} = \frac{-2}{2(3+2-1)} = \frac{-2}{8} = -\frac{1}{4}$. Tangent line approximation: $y \approx -1 + \left(-\frac{1}{4}\right)(1.6-2) = -1 - \frac{1}{4}(-0.4) = -1 + 0.1 = -0.9$. Point 3: correct slope of the tangent line at $(2,-1)$, i.e. $\left.\frac{dy}{dx}\right|_{(x,y)=(2,-1)} = -\frac{1}{4}$ (or equivalent statement, e.g. "slope is $-\frac{1}{4}$"). Point 4: correct tangent-line approximation $y \approx -0.9$, using $x = 1.6$; requires clearly demonstrated use of the

Solution 6

tangent line at $x = 1.6$ (does not require the tangent line equation to be explicitly presented).

Solution 6

(c) Mark scheme

- The curve has a vertical tangent where the denominator of $\dfrac{dy}{dx}$ is zero: $2(3y^2-2y-1)=0 \Rightarrow 2(3y+1)(y-1)=0 \Rightarrow y=-\dfrac{1}{3}$ or $y=1$. Since $y>0$ is required, $y=1$ is the y-coordinate of point S. Point 5: sets the denominator $2(3y^2-2y-1)=0$ (or an equivalent factored/unfactored form, e.g. $3y^2-2y-1=0$ or $(3y\pm 1)(y\pm 1)=0$; the numerator need not be considered). Point 6: correct answer $y=1$, with correct algebraic work clearly identifying it as the only solution satisfying $y>0$ (stating both $y=-\frac{1}{3}$ and $y=1$ without identifying $y=1$ as the answer to the prompt does not earn this point); requires Point 5.

Solution 6

(d) Mark scheme

- Differentiate $2xy + \ln y = 8$ implicitly with respect to t: $2\frac{dx}{dt}y + 2x\frac{dy}{dt} + \frac{1}{y}\frac{dy}{dt} = 0.$
  At $(x, y) = (4, 1)$ with $\frac{dx}{dt} = 3$:
  $2(3)(1) + 2(4)\frac{dy}{dt} + \frac{1}{1}\frac{dy}{dt} = 0 \Rightarrow 6 + 9\frac{dy}{dt} = 0 \Rightarrow \frac{dy}{dt} = -\frac{2}{3}$. Point 7: attempts implicit differentiation of $2xy + \ln y = 8$ with respect to t (at most one error; alternatively, differentiating with respect to x first is acceptable as long as $\frac{dy}{dx}$ is eventually correctly linked to $\frac{dy}{dt}$ via the chain rule). Point 8: correct equation

Solution 6

$2\frac{dx}{dt}y + 2x\frac{dy}{dt} + \frac{1}{y}\frac{dy}{dt} = 0$ (or equivalent); requires Point 7 with no errors in the implicit differentiation. Point 9: correct answer $\frac{dy}{dt} = -\frac{2}{3}$; requires Points 7 and 8.

Question 5

2024 ▪ Q5

Consider the curve defined by the equation $x^2 + 3y + 2y^2 = 48$. It can be shown that $\dfrac{dy}{dx} = \dfrac{-2x}{3 + 4y}$.

(a) There is a point on the curve near $(2, 4)$ with x-coordinate 3. Use the line tangent to the curve at $(2, 4)$ to approximate the y-coordinate of this point.

(b) Is the horizontal line $y = 1$ tangent to the curve? Give a reason for your answer.

(c) The curve intersects the positive x-axis at the point $(\sqrt{48}, 0)$. Is the line tangent to the curve at this point vertical? Give a reason for your answer.

(d) For time $t \geq 0$, a particle is moving along another curve defined by the equation $y^3 + 2xy = 24$. At the instant the particle is at the point $(4, 2)$, the y-coordinate of the particle's position is decreasing at a rate of 2 units per second. At that instant, what is the rate of change of the x-coordinate of the particle's position with respect to time?

Solution 5

(a) Mark scheme

- $\left.\dfrac{dy}{dx}\right|_{(2,4)} = \dfrac{-2(2)}{3+4(4)} = -\dfrac{4}{19}$. Tangent line approximation: $y \approx 4 - \dfrac{4}{19}(3-2) = \dfrac{72}{19}$. 1 point for the correct slope $-4/19$ at $(2,4)$, 1 point for the correct approximation using that slope evaluated at $x = 3$ through the point $(2,4)$.

Solution 5

(b) Mark scheme

- $\dfrac{dy}{dx} = \dfrac{-2x}{3+4y} = 0 \Rightarrow x = 0$, so if $y = 1$ were tangent, the point of tangency would have to be $(0,1)$. But $(0,1)$ is not on the curve, because $0^2 + 3(1) + 2(1)^2 = 5 \neq 48$. Therefore the horizontal line $y = 1$ is NOT tangent to the curve. 1 point for considering $dy/dx = 0$ (equivalently $x = 0$, or identifying the point $(0,1)$), 1 point for the answer 'not tangent' with a valid reason (merely stating '(0,1) is not on the curve' without explanation is insufficient; must show why).

Solution 5

(c) Mark scheme

- At $(\sqrt{48}, 0)$, $\dfrac{dy}{dx} = \dfrac{-2\sqrt{48}}{3+4(0)} = \dfrac{-2\sqrt{48}}{3}$. Since the denominator $3 + 4(0) = 3 \neq 0$, the slope is defined (finite), so the tangent line at this point is NOT vertical. The single point requires clearly showing the slope of the tangent at $(\sqrt{48}, 0)$ is defined and answering 'no' (considering only that the denominator is nonzero is sufficient).

Solution 5

(d) Mark scheme

- Curve: $y^3 + 2xy = 24$. Implicit differentiation with respect to t: $3y^2\dfrac{dy}{dt} + 2x\dfrac{dy}{dt} + 2y\dfrac{dx}{dt} = 0$. At $(4, 2)$ with $\dfrac{dy}{dt} = -2$: $3(2)^2(-2) + 2(4)(-2) + 2(2)\dfrac{dx}{dt} = 0 \Rightarrow -24 - 16 + 4\dfrac{dx}{dt} = 0 \Rightarrow \dfrac{dx}{dt} = \dfrac{40}{4} = 10.$ The rate of change of the x-coordinate is 10 units per second. 1 point for attempting implicit differentiation of $y^3 + 2xy = 24$ with respect to t (at most one error), 1 point for the fully correct differentiated equation $3y^2\frac{dy}{dt} + 2x\frac{dy}{dt} + 2y\frac{dx}{dt} = 0$, 1 point for correctly using $dy/dt = -2$ (the y-coordinate is decreasing), 1 point (only earnable with the prior three, no arithmetic mistakes) for the final answer 10.

Question 3

2023 ▪ Q3

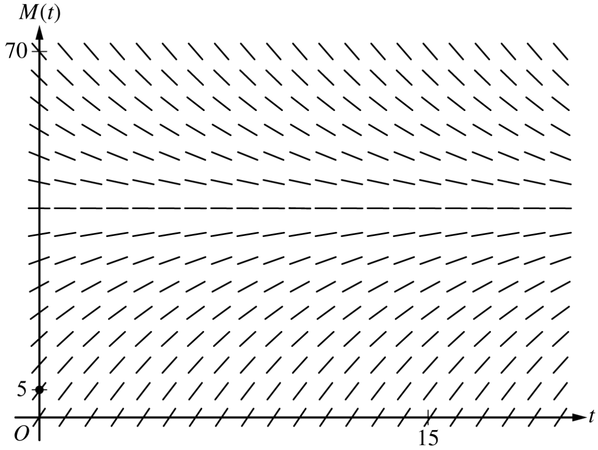
A bottle of milk is taken out of a refrigerator and placed in a pan of hot water to be warmed. The increasing function M models the temperature of the milk at time t, where $M(t)$ is measured in degrees Celsius ($^\circ$C) and t is the number of minutes since the bottle was placed in the pan. M satisfies the differential equation $\dfrac{dM}{dt} = \dfrac{1}{4}(40 - M)$. At time $t = 0$, the temperature of the milk is 5°C. It can be shown that $M(t) < 40$ for all values of t.

(a) A slope field for the differential equation $\dfrac{dM}{dt} = \dfrac{1}{4}(40 - M)$ is shown. Sketch the solution curve through the point $(0, 5)$.
[Slope field diagram: axes labeled $M(t)$ (vertical, with gridlines at 5 and 70 marked) and t (horizontal, with a gridline at 15 marked), origin O. Short line segments are

Question 3

drawn at grid points across the region $0 \leq t \leq 15$-ish, $0 \leq M(t) \leq 70$-ish: segments are nearly horizontal (flat, slightly downward-sloping) near $M(t) = 70$, becoming steeper negative slope moving down toward $M(t) \approx 40$ (segments are horizontal/flat right at $M(t) = 40$), then the slopes become positive and increasingly steep as $M(t)$ decreases from 40 down toward $M(t) = 5$, with the steepest (near-vertical, slope ≈ 1) segments at the bottom of the field near $M(t) = 5$. The point $(0, 5)$ is marked with a star on the vertical axis.]

Question 3



Question 3

2023 ▪ Q3

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(b) Use the line tangent to the graph of M at $t = 0$ to approximate $M(2)$, the temperature of the milk at time $t = 2$ minutes.

Question 3

2023 ▪ Q3

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(c) Write an expression for $\dfrac{d^2M}{dt^2}$ in terms of M. Use $\dfrac{d^2M}{dt^2}$ to determine whether the approximation from part (b) is an underestimate or an overestimate for the actual value of $M(2)$. Give a reason for your answer.

Question 3

2023 ▪ Q3

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(d) Use separation of variables to find an expression for $M(t)$, the particular solution to the differential equation $\frac{dM}{dt} = \frac{1}{4}(40 - M)$ with initial condition $M(0) = 5$.

Solution 3

(a)

Mark scheme

- The solution curve through (0,5) starts at (0,5), increases and is concave down, and approaches (curving up toward, but always staying below) the flat slope segments at M=40 as t increases across the shown window; it must not cross any drawn slope-field segments (1 pt: curve passes through (0,5), extends reasonably close to the left and right edges of the given rectangle, has no obvious conflicts with the slope lines, and lies entirely below M=40).

Solution 3

(b) Mark scheme

- $\left.\frac{dM}{dt}\right|_{t=0} = \frac{1}{4}(40-5) = \frac{35}{4}$ (1 pt: value of the derivative at t=0). Tangent line: $y = 5 + \frac{35}{4}(t-0)$, so $M(2) \approx 5 + \frac{35}{4} \cdot 2 = 22.5$; the temperature of the milk at t=2 minutes is approximately 22.5°C (1 pt: approximation using a tangent line through (0,5) with slope 35/4, correctly simplified).

Solution 3

(c) Mark scheme

- $\dfrac{d^2M}{dt^2} = -\dfrac{1}{4}\dfrac{dM}{dt} = -\dfrac{1}{4}\left(\dfrac{1}{4}(40 - M)\right) = -\dfrac{1}{16}(40 - M)$ (1 pt: correct second-derivative expression in terms of M, no subsequent simplification error).
  Since $M(t) < 40$, $\dfrac{d^2M}{dt^2} < 0$, so the graph of M is concave down; therefore the tangent-line approximation from part (b) is an overestimate (1 pt: overestimate conclusion with a reason based on concavity/decreasing dM/dt).

Solution 3

(d) Mark scheme

- Separate variables: $\dfrac{dM}{40-M} = \dfrac{1}{4}\,dt$ (1 pt: separates variables). Integrate: $\displaystyle\int \frac{dM}{40-M} = \int \frac{1}{4}\,dt \Rightarrow -\ln|40-M| = \frac{1}{4}t + C$ (1 pt: finds antiderivatives). Apply the initial condition M(0)=5: $-\ln|40-5| = 0 + C \Rightarrow C = -\ln 35$; since M(t)<40, $|40-M| = 40-M$, giving $-\ln(40-M) = \dfrac{1}{4}t - \ln 35$, i.e. $\ln(40-M) = -\dfrac{1}{4}t + \ln 35$ (1 pt: constant of integration correctly found and initial condition used). Solve for M: $40 - M = 35e^{-t/4} \Rightarrow M(t) = 40 - 35e^{-t/4}$ (1 pt: solves for M — the final answer must be exactly this closed form, or equivalent).

Question 5

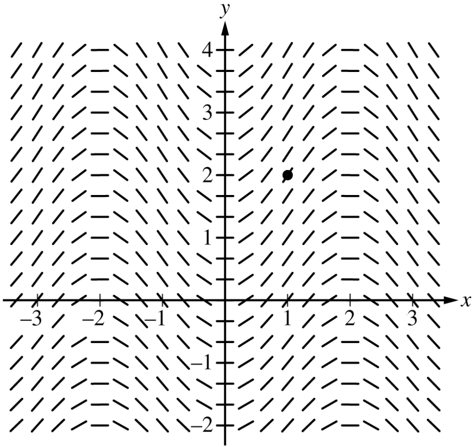
2022 ▪ Q5

Consider the differential equation $\frac{dy}{dx} = \frac{1}{2}\sin\left(\frac{\pi}{2}x\right)\sqrt{y+7}$. Let $y = f(x)$ be the particular solution to the differential equation with the initial condition $f(1) = 2$. The function f is defined for all real numbers.

(a) A portion of the slope field for the differential equation is given below. Sketch the solution curve through the point $(1, 2)$.

[Graph: a slope field on the region $-3 \leq x \leq 3$, $-2 \leq y \leq 4$ (integer gridlines on both axes). Short line segments at each grid point show the local slope; segments in the left portion (roughly $x < 0$) slant like "/" (positive slope, forward slashes), and segments in the right portion (roughly $x > 0$) slant like "(negative slope, back slashes), with a marked point at $(1, 2)$ where the solution curve should be sketched.]

Question 5



Question 5

2022 ▪ Q5

Consider the differential equation $\dfrac{dy}{dx} = \dfrac{1}{2}\sin\left(\dfrac{\pi}{2}x\right)\sqrt{y+7}$. Let $y = f(x)$ be the particular solution to the differential equation with the initial condition $f(1) = 2$. The function f is defined for all real numbers.

(b) Write an equation for the line tangent to the solution curve in part (a) at the point $(1, 2)$. Use the equation to approximate $f(0.8)$.

Question 5

2022 ▪ Q5

Consider the differential equation $\frac{dy}{dx} = \frac{1}{2}\sin\left(\frac{\pi}{2}x\right)\sqrt{y+7}$. Let $y = f(x)$ be the particular solution to the differential equation with the initial condition $f(1) = 2$. The function f is defined for all real numbers.

(c) It is known that $f''(x) > 0$ for $-1 \leq x \leq 1$. Is the approximation found in part (b) an overestimate or an underestimate for $f(0.8)$? Give a reason for your answer.

Question 5

2022 ▪ Q5

Consider the differential equation $\dfrac{dy}{dx} = \dfrac{1}{2}\sin\left(\dfrac{\pi}{2}x\right)\sqrt{y+7}$. Let $y = f(x)$ be the particular solution to the differential equation with the initial condition $f(1) = 2$. The function f is defined for all real numbers.

(d) Use separation of variables to find $y = f(x)$, the particular solution to the differential equation

$$\frac{dy}{dx} = \frac{1}{2}\sin\left(\frac{\pi}{2}x\right)\sqrt{y+7}$$

with the initial condition $f(1) = 2$.

Solution 5

(a)

Mark scheme

- The solution curve must pass through $(1, 2)$, follow the given slope field, satisfy $f(x) > 0$ at every point on the curve, extend reasonably close to the left and right edges of the given square, and have any local maximum/minimum occur only where the slope field shows horizontal segments. This is a graphical sketch, not an algebraic answer; 1 point is earned for a curve satisfying these properties.

Solution 5

(b)

Mark scheme

- $\left.\dfrac{dy}{dx}\right|_{(x,y)=(1,2)} = \dfrac{1}{2}\sin\left(\dfrac{\pi}{2}\right)\sqrt{2+7} = \dfrac{1}{2}\cdot 1\cdot 3 = \dfrac{3}{2}$. Tangent line: $y = 2 + \dfrac{3}{2}(x-1)$. Approximation: $f(0.8) \approx 2 + \dfrac{3}{2}(0.8-1) = 1.7$. Points: (1) correct tangent line equation (any equivalent form); (2) correct approximation $f(0.8) \approx 1.7$.

Solution 5

(c)

Mark scheme

- Because $f''(x) > 0$ on $-1 \leq x \leq 1$, f is concave up there, so the tangent line lies below the graph of $y = f(x)$ at $x = 0.8$, and the approximation for $f(0.8)$ found in part (b) is an underestimate. 1 point for the correct answer with a reason that references $f''(x) > 0$, f' increasing, or f concave up.

Solution 5

(d) Mark scheme

- $\dfrac{dy}{dx} = \dfrac{1}{2}\sin\left(\dfrac{\pi}{2}x\right)\sqrt{y+7} \Rightarrow \displaystyle\int \frac{dy}{\sqrt{y+7}} = \int \frac{1}{2}\sin\left(\frac{\pi}{2}x\right)\,dx$. Integrating: $2\sqrt{y+7} = -\dfrac{1}{\pi}\cos\left(\dfrac{\pi}{2}x\right) + C$. Using $f(1) = 2$: $2\sqrt{2+7} = -\dfrac{1}{\pi}\cos\left(\dfrac{\pi}{2}\right) + C \Rightarrow 6 = 0 + C \Rightarrow C = 6$. So $\sqrt{y+7} = 3 - \dfrac{1}{2\pi}\cos\left(\dfrac{\pi}{2}x\right)$, and $y = \left(3 - \dfrac{1}{2\pi}\cos\left(\dfrac{\pi}{2}x\right)\right)^2 - 7$. Points: (1) correct separation of variables; (2) one correct antiderivative; (3) the other correct antiderivative; (4) correct constant of integration found using the initial condition; (5) correctly solves for y.

Question 6

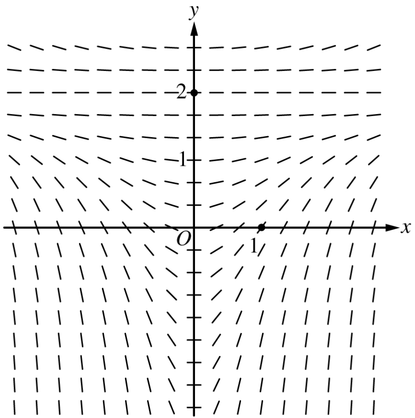
2018 ▪ Q6

Consider the differential equation $\dfrac{dy}{dx} = \dfrac{1}{3}x(y-2)^2$.

(a) A slope field for the given differential equation is shown below. Sketch the solution curve that passes through the point $(0, 2)$, and sketch the solution curve that passes through the point $(1, 0)$.

[Slope field for $\frac{dy}{dx} = \frac{1}{3}x(y-2)^2$ on axes with x-axis labeled O, 1 and y-axis labeled -1, 1, 2. For $y > 2$ and $x < 0$, the segments slope downward (negative); for $y > 2$ and $x > 0$, the segments slope upward steeply (positive) except flattening to horizontal exactly at $y = 2$ for all x; below $y = 2$ the segments are nearly horizontal near $y = 2$ and steepen (positive slope for $x > 0$, negative slope for $x < 0$) further from $y = 2$.]

Question 6



Question 6

2018 ▪ Q6

Consider the differential equation $\frac{dy}{dx} = \frac{1}{3}x(y-2)^2$.

(b) Let $y = f(x)$ be the particular solution to the given differential equation with initial condition $f(1) = 0$. Write an equation for the line tangent to the graph of $y = f(x)$ at $x = 1$. Use your equation to approximate $f(0.7)$.

Question 6

2018 ▪ Q6

Consider the differential equation $\dfrac{dy}{dx} = \dfrac{1}{3}x(y-2)^2$.

(c) Find the particular solution $y = f(x)$ to the given differential equation with initial condition $f(1) = 0$.

Solution 6

(a) Mark scheme

- Sketch two solution curves on the given slope field for $\frac{dy}{dx} = \frac{1}{3}x(y-2)^2$: one through $(0,2)$ that follows the slope lines and stays essentially flat along $y=2$ (since $y=2$ is an equilibrium solution, slope 0 there) extending to the boundary of the field; and one through $(1,0)$ that dips down following the field (roughly through the region near the origin) then curves back up toward $y=2$ as x increases, also extending to the boundary. (1 pt: correct curve through $(0,2)$; 1 pt: correct curve through $(1,0)$. Both curves must go through the indicated points, follow the given slope lines, and extend to the boundary of the slope field.)

Solution 6

(b)

Mark scheme

- $\left.\dfrac{dy}{dx}\right|_{(x,y)=(1,0)} = \frac{1}{3}(1)(0-2)^2 = \frac{4}{3}$. An equation for the line tangent to the graph of $y = f(x)$ at $x = 1$ is $y = \frac{4}{3}(x-1)$. Using this tangent line, $f(0.7) \approx \frac{4}{3}(0.7-1) = -0.4$. (1 pt: correct equation of the tangent line; 1 pt: the approximation $f(0.7) \approx -0.4$.)

Solution 6

(c) Mark scheme

- Separate variables: $\dfrac{dy}{dx} = \frac{1}{3}x(y-2)^2 \Rightarrow \int \dfrac{dy}{(y-2)^2} = \int \frac{1}{3}x\,dx \Rightarrow \dfrac{-1}{y-2} = \frac{1}{6}x^2 + C$. Using $f(1) = 0$: $\frac{1}{2} = \frac{1}{6} + C \Rightarrow C = \frac{1}{3}$. So $\dfrac{-1}{y-2} = \frac{1}{6}x^2 + \frac{1}{3} = \dfrac{x^2+2}{6}$, giving $y = 2 - \dfrac{6}{x^2+2}$ (valid for $-\infty < x < \infty$). (1 pt: separates variables; 2 pts: correct antiderivatives on both sides; 1 pt: constant of integration found using the initial condition; 1 pt: solves explicitly for y. Note: 0/5 if no separation of variables; max 3/5 if no constant of integration is found/used.)

Question 4

2017 ▪ Q4

At time $t = 0$, a boiled potato is taken from a pot on a stove and left to cool in a kitchen. The internal temperature of the potato is 91 degrees Celsius ($^\circ C$) at time $t = 0$, and the internal temperature of the potato is greater than $27^\circ C$ for all times $t > 0$. The internal temperature of the potato at time t minutes can be modeled by the function H that satisfies the differential equation $\frac{dH}{dt} = -\frac{1}{4}(H - 27)$, where $H(t)$ is measured in degrees Celsius and $H(0) = 91$.

(a) Write an equation for the line tangent to the graph of H at $t = 0$. Use this equation to approximate the internal temperature of the potato at time $t = 3$.

(b) Use $\frac{d^2H}{dt^2}$ to determine whether your answer in part (a) is an underestimate or an overestimate of the internal temperature of the potato at time $t = 3$.

Question 4

(c) For $t < 10$, an alternate model for the internal temperature of the potato at time t minutes is the function G that satisfies the differential equation $\dfrac{dG}{dt} = -(G - 27)^{2/3}$, where $G(t)$ is measured in degrees Celsius and $G(0) = 91$. Find an expression for $G(t)$. Based on this model, what is the internal temperature of the potato at time $t = 3$?

Question 4

2016 ▪ Q4

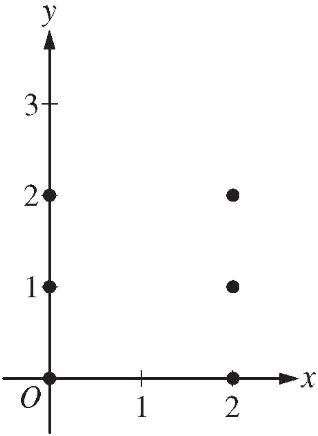
Consider the differential equation $\frac{dy}{dx} = \frac{y^2}{x-1}$.

(a) On the axes provided, sketch a slope field for the given differential equation at the six points indicated.
[Slope-field axes: x-axis labeled with tick marks at 1 and 2, y-axis labeled with tick marks at 1, 2, 3; the six indicated points (dots) are at $(0,0)$, $(0,1)$, $(0,2)$, $(2,0)$, $(2,1)$, $(2,2)$.]

(b) Let $y = f(x)$ be the particular solution to the given differential equation with the initial condition $f(2) = 3$. Write an equation for the line tangent to the graph of $y = f(x)$ at $x = 2$. Use your equation to approximate $f(2.1)$.

(c) Find the particular solution $y = f(x)$ to the given differential equation with the initial condition $f(2) = 3$.

Question 4



Solution 4

(a) Mark scheme

- Slope field at the six indicated points, using dy/dx = y^2/(x-1): at (0,0): slope = 0^2/(0-1)=0; at (2,0): slope = 0^2/(2-1)=0 — these are the ZERO-slope points (horizontal segments). At (0,1): slope = 1^2/(0-1)=-1; at (0,2): slope = 2^2/(0-1)=-4; at (2,1): slope = 1^2/(2-1)=1; at (2,2): slope = 2^2/(2-1)=4 — these are the NONZERO-slope points, drawn as short segments with the corresponding sign and relative steepness. Points: 1 for correctly drawing zero slopes at (0,0) and (2,0), 1 for correctly drawing nonzero slopes (correct sign and relative steepness) at the other four points.

Solution 4

(b)

Mark scheme

- $\frac{dy}{dx}\Big|_{(x,y)=(2,3)} = \frac{3^2}{2-1} = 9. Tangentline : y = 9(x-2)+3. Approximation : f(2.1) \approx 9(2.1-2)+3 = 9(0.1)+3 = 3.9. Points : 1 forthecorrecttangentlineequation, 1 forthecorrectapproximationf(2.1) \approx 3.9.$

Solution 4

(c) Mark scheme

- Separate variables: $\frac{1}{y^2}\,dy = \frac{1}{x-1}\,dx. Integrate: \int \frac{1}{y^2}\,dy = \int \frac{1}{x-1}\,dx \Rightarrow -\frac{1}{y} = \ln|x-1| + C. Applytheinitialconditionf(2) = 3 : -\frac{1}{3} = \ln|2-1| + C = 0 + C \Rightarrow C = -\frac{1}{3}. So -\frac{1}{y} = \ln|x-1| - \frac{1}{3}. Solvefory : y = \frac{1}{\frac{1}{3} - \ln(x-1)}(validfor 1 < x < 1 + e^{1/3}). Points : 1 forseparatingvariables, 2 forcorrectantiderivatives(1 foreachside), 1 forincludingtheconstantofint$
  $max 3/5 ifnoconstantofintegrationisincluded; 0/5 ifvariablesarenotseparated.$

Question 1

2014 ▪ Q1

Grass clippings are placed in a bin, where they decompose. For $0 \le t \le 30$, the amount of grass clippings remaining in the bin is modeled by $A(t) = 6.687(0.931)^t$, where $A(t)$ is measured in pounds and t is measured in days.

(a) Find the average rate of change of $A(t)$ over the interval $0 \le t \le 30$. Indicate units of measure.

(b) Find the value of $A'(15)$. Using correct units, interpret the meaning of the value in the context of the problem.

(c) Find the time t for which the amount of grass clippings in the bin is equal to the average amount of grass clippings in the bin over the interval $0 \le t \le 30$.

(d) For $t > 30$, $L(t)$, the linear approximation to A at $t = 30$, is a better model for the amount of grass clippings remaining in the bin. Use $L(t)$ to predict the time at which

Question 1

there will be 0.5 pound of grass clippings remaining in the bin. Show the work that leads to your answer.

Solution 1

(a)

Mark scheme

- Average rate of change = [A(30) - A(0)] / (30 - 0) = -0.197 (or -0.196) lbs/day. Points: 1 for the correct numeric value with correct units (pounds per day).

Solution 1

(b)

Mark scheme

- A'(15) = -0.164 (or -0.163) lbs/day. Interpretation: the amount of grass clippings in the bin is decreasing at a rate of 0.164 (or 0.163) lb/day at time t = 15 days. Points: 1 for the correct value of A'(15), 1 for a correct interpretation in context (decreasing, correct units, at t = 15).

Solution 1

(c)

Mark scheme

- Set A(t) equal to the average value of A over [0,30]: A(t) $= (1/30) \int_0^{30} A(t)\, dt, which gives t = 12.415 (or 12.414). Points: 1 for the correct average-value integral expression (1/30) \int_0^{30} A(t)\, dt, 1 for the correct answer t \approx 12.415.$

Solution 1

(d)

Mark scheme

- L(t) = A(30) + A'(30)·$(t - 30), with A'(30) = -0.055976 and A(30) = 0.782928. Setting L(t) = 0.5 give st = 35.054. Points: 2 for a correct expression for L(t) (linear approximation/tangent line to A at t = 30), 1 for setting L(t) = 0.5, 1 for the correct final answer t \approx 35.054.$

Question 6

2014 ▪ Q6

Consider the differential equation $\dfrac{dy}{dx} = (3 - y)\cos x$. Let $y = f(x)$ be the particular solution to the differential equation with the initial condition $f(0) = 1$. The function f is defined for all real numbers.

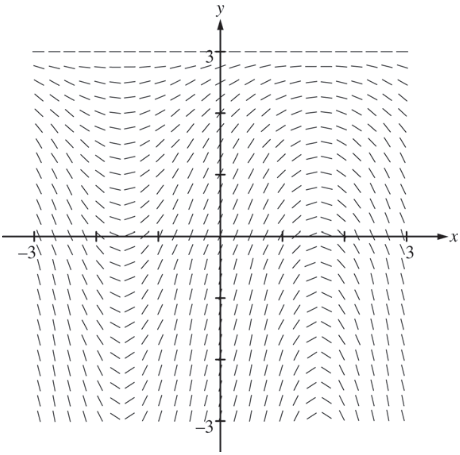
(a) A portion of the slope field of the differential equation is given below. Sketch the solution curve through the point $(0, 1)$.

[Slope field diagram on the domain $-3 \leq x \leq 3$, $-3 \leq y \leq 3$, with tick marks at -3, 1, 2(implied), 3 on both axes (visible ticks at -3 and 3 on the x-axis, at 3 and -3 on the y-axis, plus unlabelled tick marks at intermediate integer values). Short line segments show the slope at each grid point: segments are roughly horizontal (dashes) in the upper-left and upper-right regions (where y is large, slope near 0 since $3 - y$ is very negative but cos x oscillates - shown as near-flat dashes), segments tilt more

Question 6

steeply (steep near-vertical strokes) in the band around $y \approx 1$ to 2 near $x = 0$, and the pattern is symmetric in a banded way across the vertical line $x = 0$, repeating roughly periodically along the x-axis consistent with the $\cos x$ factor.]

Question 6



Question 6

2014 ▪ Q6

Consider the differential equation $\frac{dy}{dx} = (3 - y)\cos x$. Let $y = f(x)$ be the particular solution to the differential equation with the initial condition $f(0) = 1$. The function f is defined for all real numbers.

(b) Write an equation for the line tangent to the solution curve in part (a) at the point $(0, 1)$. Use the equation to approximate $f(0.2)$.

Question 6

2014 ▪ Q6

Consider the differential equation $\dfrac{dy}{dx} = (3 - y)\cos x$. Let $y = f(x)$ be the particular solution to the differential equation with the initial condition $f(0) = 1$. The function f is defined for all real numbers.

(c) Find $y = f(x)$, the particular solution to the differential equation with the initial condition $f(0) = 1$.

Solution 6

(a) Mark scheme

- Sketch the solution curve through (0,1), following the slope-field segment directions: the curve should dip to a local minimum to the left of x=0 (matching the field's near-horizontal/negative segments there) and rise through (0,1) toward a maximum further to the right, staying consistent with the plotted slope segments across -3$\leq x \leq 3 (matchestheexactsolutiony = 3 - 2e^{-\sin x}, whichhasaminimumnearx = -\pi/2andamaximumnearx = \pi/2). Points : 1foracurvethatpassesthrough(0,1)andisconsistentwiththegivenslopefield' ssegmentdirectionsthr$

Solution 6

(b)

Mark scheme

- $\frac{dy}{dx}\Big|_{(0,1)} = (3-1)\cos(0) = 2\cdot 1 = 2. Tangentline: y = 2x+1 (point-slopethrough(0,1)withslope2). Usingthetangentline, f(0.2$

Solution 6

(c) Mark scheme

- Separate variables:

$$\text{dy}\frac{}{dx = (3-y)\cos x \Rightarrow \int \frac{dy}{3-y} = \int \cos x \, dx \Rightarrow -\ln|3-y| = \sin x + C. Applytheinitialconditionf(0) = 1: -\ln|3-1| = \sin(0) + C \Rightarrow -\ln}$$