Limits and Continuity

AP Calculus AB

Introducing Calculus: Can Change Occur at an Instant?

Calculus is the mathematics of **change** and of **accumulation** . It answers two big questions: how fast is something changing *right now*, and how much has piled up *so far*? Unit 1 builds the one tool both questions rest on – the **limit** .

Start with a puzzle. A car's speedometer reads 60 km/h. What does that mean *at a single instant* ? Speed is distance over time. But at one instant no time passes and no distance is covered, so the fraction looks like $\frac{0}{0}$ – undefined.

- The **average rate of change** uses a whole *interval* : the change in one quantity divided by the change in another. It divides by zero, and so is undefined, when the change in the input would be zero.
- The **instantaneous rate of change** is what we want *at a point*. It is the value the average rate **approaches** as the interval shrinks toward zero length.

The clever move is not to plug in zero (undefined), but to watch what the average rate *approaches* as the interval gets smaller and smaller. That approaching value is a **limit**. So calculus lets us describe change at an instant – as a limit of average rates over ever-shorter intervals. This one idea powers the **derivative** (Unit 2) and, run in reverse, the **integral** (Unit 6). Everything else in this unit defines limits carefully and computes them reliably.

Defining Limits and Using Limit Notation



A smooth bridge curve: limits describe the value a graph approaches as we zoom in

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Given a function f, the limit of $f(x)$ as x approaches c is a real number R if $f(x)$ can be made **arbitrarily** close to R by taking x **sufficiently** close to c – **but not equal to** c. We write

$$\lim_{x \to c} f(x) = R$$

and read it: "the limit of $f(x)$, as x approaches c, equals R."

The last words are the heart of a limit: it describes the **behavior** of f *near* c, not the value *at* c. The function may be undefined at c, or defined but equal to something else – the limit does not care.

A limit can be shown in three ways: **graphically** , **numerically** (a table), and **analytically** (algebra). Learning to move between these representations is a core skill.

*(Note: the epsilon-delta definition of a limit is **not** tested on the AP Exam, so this handout does not use it.)*

Estimating Limit Values from Graphs

A graph is often the fastest way to read a limit. To find $\lim_{x \to c} f(x)$, run your finger along the curve toward $x = c$ from each side and ask: what height is the curve heading for?

- Trace from the **left** (inputs smaller than c): this gives the **left-hand limit** , $\lim_{x \to c^-} f(x)$.
- Trace from the **right** (inputs larger than c): this gives the **right-hand limit** , $\lim_{x \to c^+} f(x)$.
- These are the **one-sided limits** . If both head to the *same* height R, then the two-sided limit exists and $\lim_{x \to c} f(x) = R$.

Crucially, **ignore the point itself**. Graphs mark the difference between the limit and the value:

- An **open circle** marks a height the curve *approaches* but does not reach – a "hole" .
- A **closed circle** marks the actual value $f(c)$.

A limit **does not exist** (often written **DNE**) when the two sides disagree (a **jump**), when the function is **unbounded** (grows without limit), or when it **oscillates** near c with fluctuations that do not shrink toward one value. Oscillation alone does not prevent a limit: $x \sin(1/x) \to 0$ as $x \to 0$. For example:

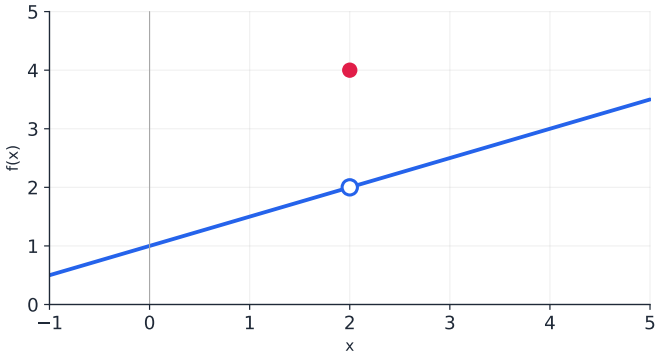
$$\lim_{x \to 0} \frac{1}{x^2} = \infty, \qquad \lim_{x \to 0} \frac{|x|}{x} \text{ DNE}, \qquad \lim_{x \to 0} \sin\left(\frac{1}{x}\right) \text{ DNE}.$$

Watch the **scale** of a graph: a zoomed-out picture can hide important behavior near a point, so confirm with algebra when you can.

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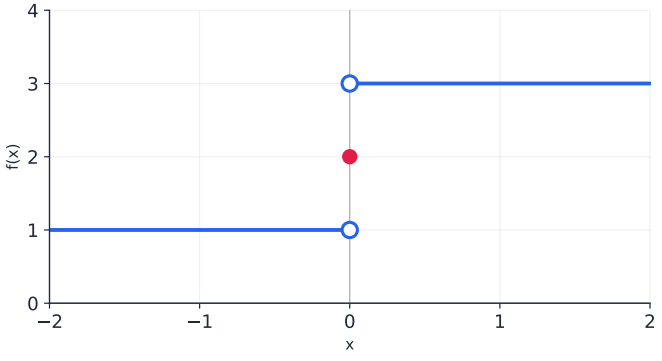
■ Worked comparison: nearby values and one point

In this teacher-created graph, read the approaching heights before the filled point.



A curve with an open point at (2,2) and a filled point at (2,4).

Both one-sided limits at 2 equal 2, so the two-sided limit is 2. The filled point gives $f(2) = 4$. This is a removable discontinuity: set $f(2) = 2$ to make the function continuous.



One-sided heights 1 and 3 at 0, with a separate filled point at height 2.

Here the left-hand limit at 0 is 1, the right-hand limit is 3, and $f(0) = 2$. The unequal one-sided limits mean the two-sided limit does not exist. No choice of $f(0)$ changes the nearby branches; the jump cannot be removed by changing one point.

Estimating Limit Values from Tables

When you have data or a formula but no picture, a **table** of values estimates a limit numerically. Choose inputs that creep toward c from both sides and watch the outputs.

For example, to estimate $\lim_{x \to 2} \dfrac{x^2 - 4}{x - 2}$ (which is $\frac{0}{0}$ at $x = 2$):

x	1.9	1.99	1.999	$\to 2 \leftarrow$	2.001	2.01	2.1
$f(x)$	3.9	3.99	3.999	?	4.001	4.01	4.1

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Both sides march toward 4, so we estimate the limit is 4. A table only *suggests* a value – it is a numerical estimate, not a proof.

x	1.9	1.99	1.999	2	2.001	2.01	2.1
$f(x)$	3.9	3.99	3.999	?	4.001	4.01	4.1

from the left: 4 — from the right: 4

both sides agree, so $\lim_{x\to 2} f(x) = 4$, whatever $f(2)$ is

Values from both sides close in on 4, so the limit is 4 whatever f(2) is

Determining Limits Using Algebraic Properties of Limits

Most limits are found **analytically** using **limit theorems** . If $\lim_{x\to c} f(x)$ and $\lim_{x\to c} g(x)$ both exist, the limit of a combination is the same combination of the limits:

- **Sum / difference:** $\lim\limits_{x\to c}\big[f(x)\pm g(x)\big] = \lim\limits_{x\to c} f(x) \pm \lim\limits_{x\to c} g(x)$
- **Product:** $\lim\limits_{x\to c}\big[f(x)\,g(x)\big] = \lim\limits_{x\to c} f(x)\cdot \lim\limits_{x\to c} g(x)$
- **Quotient:** $\lim\limits_{x\to c}\dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to c} f(x)}{\lim_{x\to c} g(x)}$, provided the bottom limit is not 0.
- **Composite** : if g is continuous at $\lim_{x\to c} f(x)$, then $\lim\limits_{x\to c} g\big(f(x)\big) = g\Big(\lim\limits_{x\to c} f(x)\Big)$.

The practical rule: for a function built from polynomials, roots, and the like, first try **direct substitution** – put $x = c$ in. If you get a real number, that is the limit. One-sided limits obey the same theorems, read from one direction only.

Determining Limits Using Algebraic Manipulation

Direct substitution sometimes gives the **indeterminate form** $\frac{0}{0}$. This does **not** mean the limit fails – it means you must rewrite the function into an **equivalent form** that removes the trouble, then substitute. Three standard moves:

- **Factor and cancel** a **rational function** . Example: $\lim\limits_{x\to 2}\dfrac{x^2-4}{x-2} = \lim\limits_{x\to 2}\dfrac{(x-2)(x+2)}{x-2} = \lim\limits_{x\to 2}(x+2) = 4.$

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- Multiply by the **conjugate** to simplify a **radical**. Example: $\lim_{x\to 0}\dfrac{\sqrt{x+1}-1}{x}=\lim_{x\to 0}\dfrac{x}{x\left(\sqrt{x+1}+1\right)}=\dfrac{1}{2}$.
- **Use alternate forms** of trigonometric functions (identities) to simplify.

The cancelled factor is why the original graph had a hole: the two functions agree everywhere except at $x=c$, so they share the same limit there.

Selecting Procedures for Determining Limits

This is a skill topic, not new content: choose the right tool for the limit in front of you.

1. **Try direct substitution first.** A real answer means you are done.
2. Getting $\frac{0}{0}$? **Rewrite** – factor and cancel, or use the conjugate, or a trig identity – then substitute.
3. A **non-zero number over** 0 (like $\frac{5}{0}$)? The limit is infinite or DNE – check the sign from each side (see vertical asymptotes below).
4. As $x\to\pm\infty$? Compare the **dominant** terms (see limits at infinity).
5. Trapped between two functions? The **squeeze theorem** may apply.

Determining Limits Using the Squeeze Theorem

The **squeeze theorem** (also called the sandwich theorem) finds a limit by trapping the function between two others. If $g(x)\le f(x)\le h(x)$ near c, and

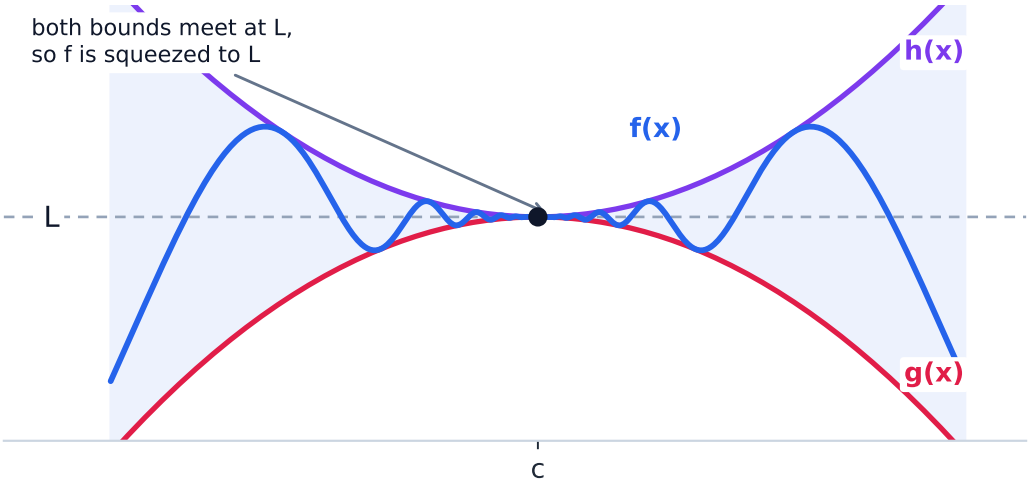
$$\lim_{x\to c}g(x)=\lim_{x\to c}h(x)=L,$$

then f is squeezed to the same place: $\lim_{x\to c}f(x)=L$.

The two famous results proved this way, both used throughout calculus, are:

$$\lim_{x\to 0}\frac{\sin x}{x}=1 \qquad \text{and} \qquad \lim_{x\to 0}\frac{1-\cos x}{x}=0.$$

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if $g \le f \le h$ near c and $\lim g = \lim h = L$, then $\lim f = L$

The wiggly f is trapped between g and h. Because both bounds meet at L, f has nowhere to go but L.

Connecting Multiple Representations of Limits

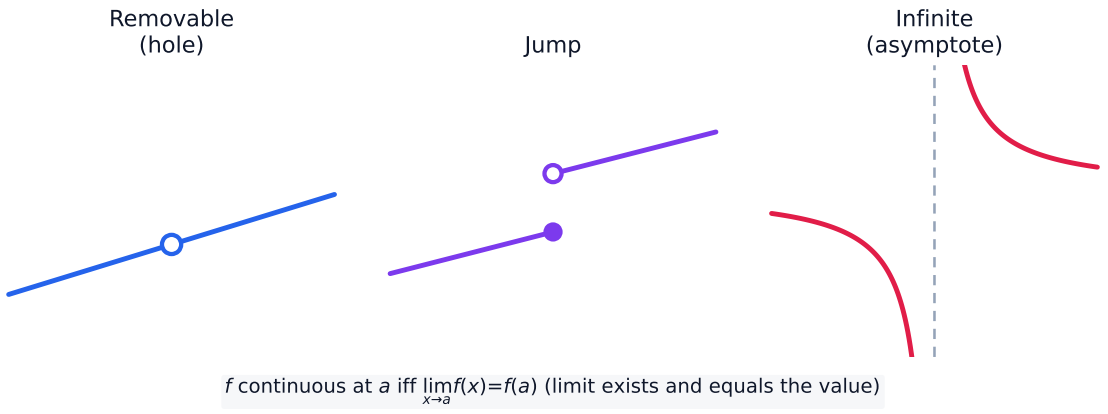
Another skill topic: the same limit lives in a graph, a table, and an algebraic form, and you should be able to translate between them. A graph shows the shape and any holes or jumps; a table gives numerical evidence; algebra gives an exact value and a reason. Strong answers use one representation to *confirm* another.

Exploring Types of Discontinuities

A function is **discontinuous** at c when its graph "breaks" there. There are three types:

- **Removable discontinuity** – a single hole. The two-sided limit exists, but the point is missing or misplaced.
- **Jump discontinuity** – the two one-sided limits exist but disagree, so the curve jumps.
- **Infinite discontinuity** – the function blows up to $\pm\infty$ at a **vertical asymptote** .

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Three ways a graph can break: a removable hole, a jump, and an infinite (asymptote) discontinuity.

■ **Worked comparison: classify and justify a repair**

At 2, $A(x) = (x^2 - 4)/(x - 2)$ simplifies to $x + 2$ for $x \neq 2$. Its limit is 4; define $A(2) = 4$ to remove the hole. In contrast, $B(x) = 1$ for $x < 2$ and 3 for $x \geq 2$ has unequal finite one-sided limits 1 and 3. Changing $B(2)$ cannot remove its jump. For $C(x) = 1/(x - 2)$, the left side tends to $-\infty$ and the right to $+\infty$: an infinite discontinuity, with no finite limit to use as a repair.

Defining Continuity at a Point

Continuity is defined by a three-part test. A function f is **continuous** at $x = c$ exactly when all three hold:

$$f(c) \text{ exists} \quad \text{and} \quad \lim_{x \to c} f(x) \text{ exists} \quad \text{and} \quad \lim_{x \to c} f(x) = f(c)$$

In words: the point is there, the limit is there, and the two agree. If any one fails, f is discontinuous at c. This test is the backbone of nearly every continuity question, so learn it as a checklist.

Confirming Continuity over an Interval

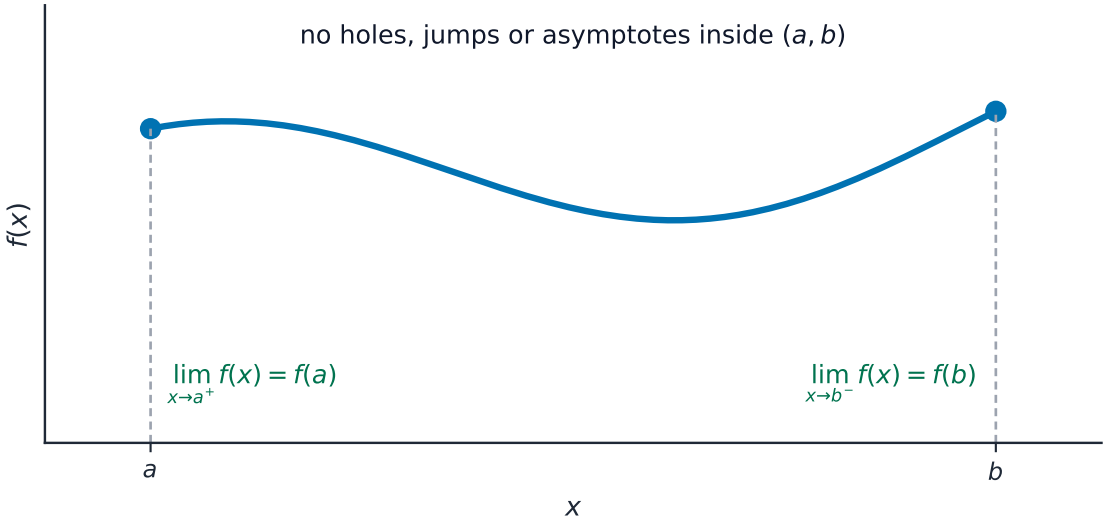
A function is **continuous on an interval** if it is continuous at *every* point of that interval. You rarely check point by point, because whole families are continuous on their domains:

> **Polynomial, rational, power, exponential, logarithmic, and trigonometric functions are continuous at every point of their domains.**

So a rational function is continuous everywhere *except* where its denominator is zero; $\ln x$ is continuous for $x > 0$; and so on. Knowing this lets you declare continuity

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quickly and correctly.



Continuous on [a, b]: no breaks inside and the one-sided limits match at the ends

Removing Discontinuities

If the **limit exists** at a hole, the discontinuity is **removable**: redefine the function at that one point to equal the limit, and the graph is repaired. Formally, set the missing value to $\lim_{x \to c} f(x)$.

For a **piecewise-defined function** , continuity at a boundary $x = c$ needs the two pieces to *meet*: the left piece's value, the right piece's value, and $f(c)$ must all be equal. This is a common exam setup – you solve for a **parameter** (an unknown constant) that makes the pieces match:

$$\lim_{x \to c^-} f(x) = \lim_{x \to c^+} f(x) = f(c).$$

Worked example. Find the constant k that makes this function continuous at $x = 2$:

$$f(x) = \begin{cases} kx + 1, & x < 2 \\ x^2 - k, & x \geq 2 \end{cases}$$

- Left-hand limit: $\lim_{x \to 2^-} (kx + 1) = 2k + 1$.
- Right-hand limit and value: $\lim_{x \to 2^+} (x^2 - k) = f(2) = 4 - k$.
- Set them equal: $2k + 1 = 4 - k$, so $3k = 3$ and $k = 1$.

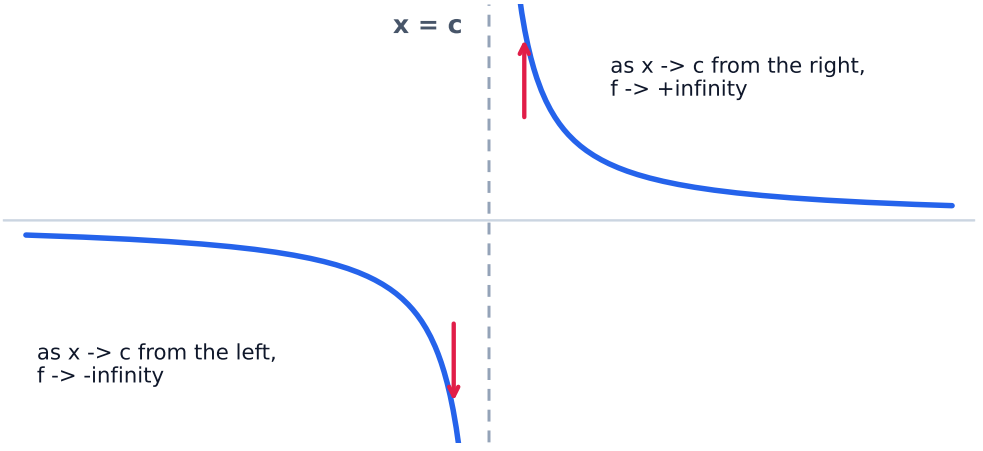
Check: with $k = 1$ both pieces give 3 at $x = 2$, so all three conditions of the continuity test hold.

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Connecting Infinite Limits and Vertical Asymptotes

The idea of a limit extends to **infinite limits** . When a function grows without bound near $x = c$, we write $\lim_{x\to c} f(x) = \pm\infty$. This describes a **vertical asymptote** at $x = c$: the graph hugs the vertical line $x = c$ and shoots off toward $\pm\infty$.

This happens where a **non-zero number is divided by something approaching** 0, such as at a zero of a denominator that does not cancel. Always check each side separately – the two sides can shoot **opposite** ways (one to $+\infty$, one to $-\infty$).



vertical asymptote $x = c$ when $\lim_{x\to c^+} f(x) = \pm\infty$ or $\lim_{x\to c^-} f(x) = \pm\infty$

Near a vertical asymptote the graph hugs the line $x = c$. Here the left side falls to $-\infty$ and the right side climbs to $+\infty$, so the two-sided limit does not exist.

Worked example. Find $\lim_{x\to 1^-} \dfrac{x+2}{x-1}$ and $\lim_{x\to 1^+} \dfrac{x+2}{x-1}$.

Substitution gives $\frac{3}{0}$: a non-zero number over 0, so each one-sided limit is infinite. Decide the sign from each side:

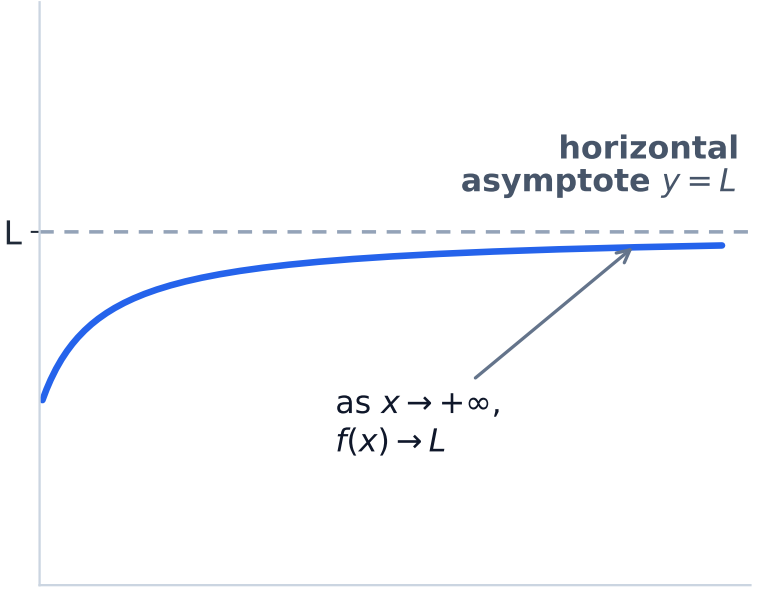
- For x just below 1 (try $x = 0.99$): top ≈ 3, bottom $= -0.01$, a small negative number. So $\lim_{x\to 1^-} \dfrac{x+2}{x-1} = -\infty$.
- For x just above 1 (try $x = 1.01$): top ≈ 3, bottom $= +0.01$. So $\lim_{x\to 1^+} \dfrac{x+2}{x-1} = +\infty$.

The line $x = 1$ is a vertical asymptote, and the two-sided limit does not exist.

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Connecting Limits at Infinity and Horizontal Asymptotes

We can also let the **input** grow: **limits at infinity** describe the **end behavior** of a function as $x \to \pm\infty$. If the outputs settle toward a finite value L, then $y = L$ is a **horizontal asymptote** .



As $x \to +\infty$ the curve flattens toward the line $y = L$. The horizontal asymptote records this end behavior.

For a rational function, compare the **degrees** of the top and bottom:

- top degree $<$ bottom degree $\Rightarrow$ limit is 0 (asymptote $y = 0$);
- top degree $=$ bottom degree $\Rightarrow$ limit is the **ratio of the leading coefficients** ;
- top degree $>$ bottom degree $\Rightarrow$ the function is unbounded (no horizontal asymptote).

Worked example. Find $\lim_{x\to\infty} \dfrac{3x^2 - 5x}{6x^2 + 1}$.

The degrees are equal (2 and 2), so the limit is the ratio of the leading coefficients. To show why, divide every term by x^2:

$$\lim_{x\to\infty} \frac{3x^2 - 5x}{6x^2 + 1} = \lim_{x\to\infty} \frac{3 - \frac{5}{x}}{6 + \frac{1}{x^2}} = \frac{3 - 0}{6 + 0} = \frac{1}{2}.$$

So $y = \frac{1}{2}$ is a horizontal asymptote.

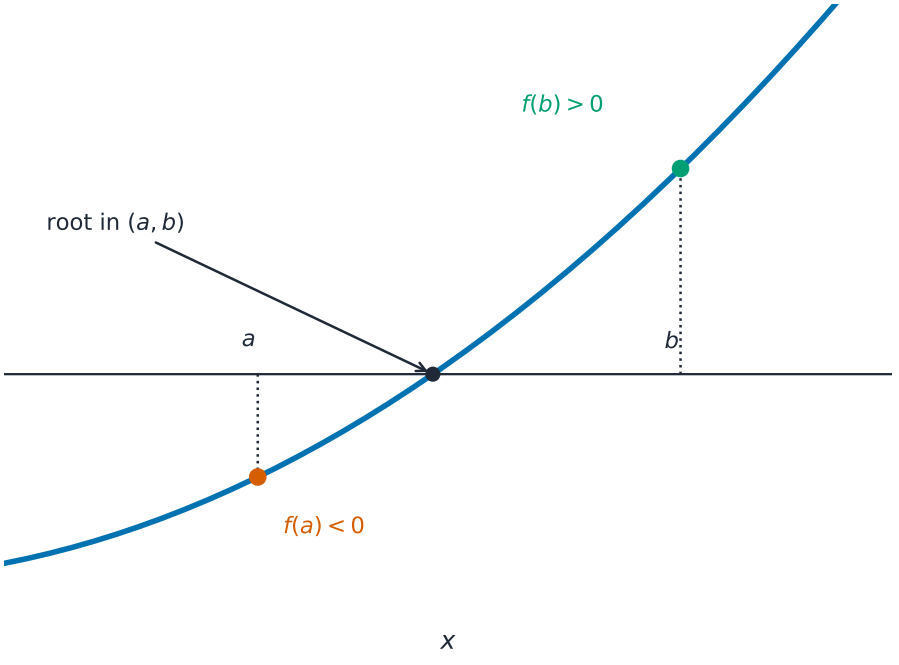
More generally, we compare the **relative magnitudes** (relative growth rates) of functions: far out, an exponential beats any polynomial, and a polynomial

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beats any logarithm. On the exam, "as $t \to \infty$, which quantity is larger/where does the rate settle?" is answered with a limit at infinity.

Working with the Intermediate Value Theorem (IVT)

The **Intermediate Value Theorem** is an **existence theorem** – it guarantees a value exists without telling you where:

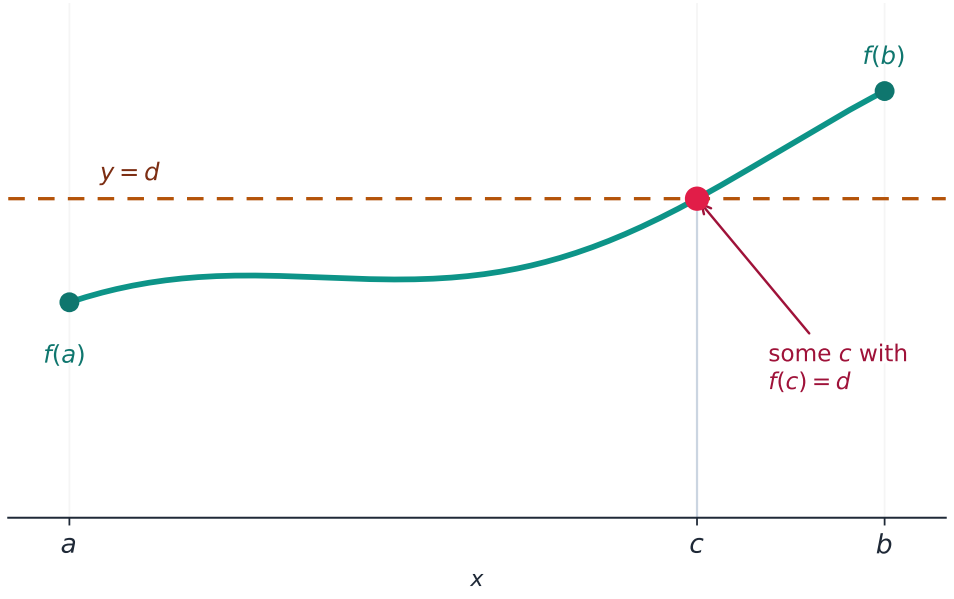


Opposite signs of f(a) and f(b) trap a root between a and b

If f is **continuous** on the closed interval $[a, b]$, and d is any number **between** $f(a)$ and $f(b)$, then there is **at least one** number c in (a, b) with $f(c) = d$.

An unbroken curve cannot skip a height between its endpoints – it must pass through every one.

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A continuous curve from $(a, f(a))$ to $(b, f(b))$ must cross every height d in between at least once.

Exam skill – how to justify with the IVT. These questions appear almost every year (for example, "Must there be a value c with $R(c) = 155$?" or "Is there a time when $r'(t) = -6$?"). A full-credit justification has three moves:

1. **State continuity.** Say the function is continuous on $[a, b]$ (often because it is differentiable, or given continuous).
2. **Show d is trapped.** Compute the two endpoint values and show the target d lies **between** them, e.g. $f(a) < d < f(b)$.
3. **Conclude by name.** "By the Intermediate Value Theorem, there is a c in (a, b) with $f(c) = d$."

Skipping the continuity statement, or not showing d is between the endpoints, loses the point – the theorem *requires* both conditions.

> **Worked example.** Show that $f(x) = x^3 - x - 1$ has a zero between $x = 1$ and $x = 2$.

1. f is a polynomial, so it is continuous on $[1, 2]$.
2. $f(1) = 1 - 1 - 1 = -1$ and $f(2) = 8 - 2 - 1 = 5$, so $f(1) < 0 < f(2)$.
3. By the Intermediate Value Theorem, there is a c in $(1, 2)$ with $f(c) = 0$.

The theorem says nothing about *where* c is or how many such values there are (here $c \approx 1.32$, found with a calculator).

Common mistakes with the IVT.

- Using it on a function that is not continuous on the whole interval: $g(x) = \frac{1}{x}$ has $g(-1) = -1$ and $g(1) = 1$, but g is never 0, because g breaks at $x = 0$.

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- Reading it backwards: a continuous f can reach a height *outside* $[f(a), f(b)]$ too, and can reach d more than once. The theorem only promises *at least one* c for each d between the endpoint values.

Exam tips

- A limit describes what $f(x)$ **approaches**, which need not equal $f(a)$ – the two-sided limit exists only if both sides agree.
- Try direct substitution first; for a $\frac{0}{0}$ form, **factor and cancel** or rationalise before substituting.
- A function is **continuous** at a when the limit exists, $f(a)$ is defined, and they are equal.
- Use the **Intermediate Value Theorem** to guarantee a root: a continuous function that changes sign on $[a, b]$ takes every value between.
- Read horizontal asymptotes from end behaviour (limits at $\pm\infty$) and vertical asymptotes at denominator zeros that remain after common factors are cancelled; check the one-sided limits.

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