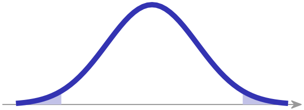
Probability & Statistics 2

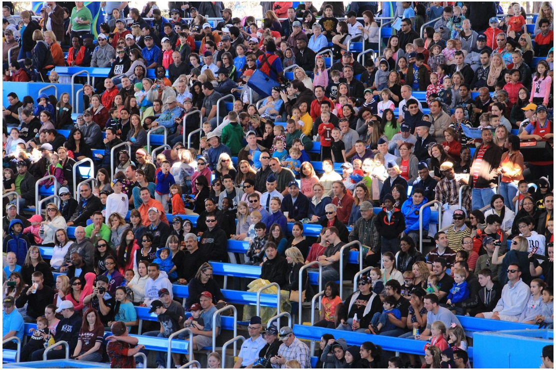
A-Level Mathematics ▪ Topic 6

A-Level Mathematics



What we will cover

1. The Poisson distribution
2. Combining random variables
3. Continuous random variables
4. Sampling & estimation
5. Hypothesis tests
6. Recap



1 Poisson and combinations

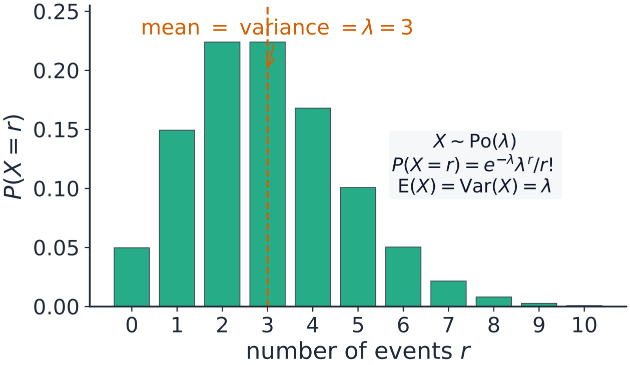
The Poisson distribution

Counts of rare, independent events in a fixed interval: $X \sim \mathrm{Po}(\lambda)$,

$$\mathrm{P}(X = k) = e^{-\lambda}\frac{\lambda^k}{k!}, \qquad \mathrm{E}(X) = \mathrm{Var}(X) = \lambda.$$

> A binomial $X \sim \mathrm{B}(n, p)$ with n large and p small is approximately Poisson with $\lambda = np$.

The Poisson distribution, in detail



The Poisson distribution $\mathrm{Po}(3)$: for a Poisson variable the mean and variance both equ

Combining random variables

For a linear change and for independent variables:

$$E(aX + b) = aE(X) + b$$

$$\mathrm{Var}(aX+bY) = a^2\mathrm{Var}X + b^2\mathrm{Var}Y$$

e.g. mean 5, variance 4: $E(3X - 1) = 14$, $\mathrm{Var}(3X - 1) = 36$.

2 Continuous and sampling

Continuous random variables

A **density function** 概率密度函数 $f(x) \geqslant 0$, $\int f = 1$.

$$P(a<X<b) = \int_a^b f, \quad E(X) = \int xf.$$

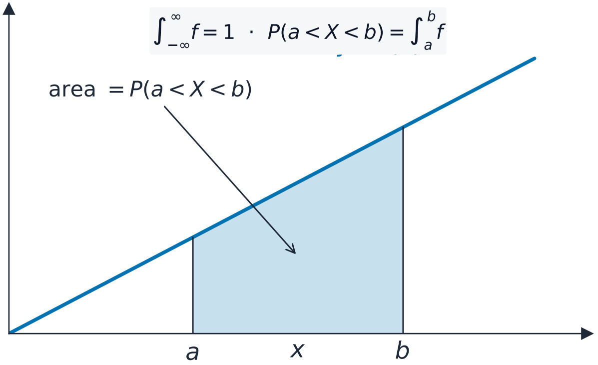
Median 中位数 solves $F(m) = 0.5$.

The two functions of a continuous variable

Probability density function	The **probability density function** 概率密度函数 $f(x)$: probability is the *area* beneath it, and the total area is 1.
Cumulative distribution function	The **cumulative distribution function** 累积分布函数 $F(x) = \mathrm{P}(X \leqslant x)$ – the integral of f, and differentiating F gives f back.
Expectation	The **expectation** 期望 is $\mathrm{E}(X) = \int xf(x)\,dx$ over the range where f is non-zero.
The normal distribution	The **normal distribution** 正态分布 is the case whose density is that symmetric bell – and the one every table is written for.

Medians and **percentiles** 百分位数 come from F, means and variances from f. Knowing which one a question needs saves most of the work.

Continuous random variables, in detail



A **density function** 概率密度函数 is never negative and encloses unit area: $f(x) \geqslant 0$ and $\int f = 1$. Probability is an area, so $P(X = a) = 0$ for any single value.

Sampling and estimation

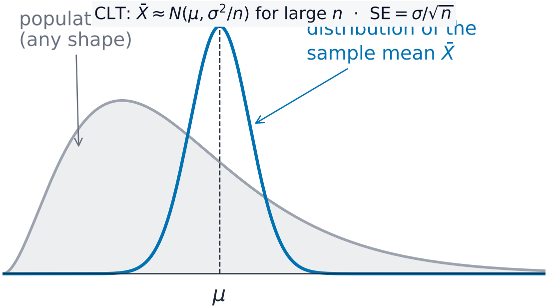
$E(\bar{X}) = \mu$, $\mathrm{Var}(\bar{X}) = \dfrac{\sigma^2}{n}$. A 95% **confidence interval** 置信区间:

$$\bar{x} \pm 1.96 \frac{\sigma}{\sqrt{n}}.$$

$n = 64$, $\bar{x} = 50$, $\sigma = 8$

$50 \pm 1.96 \Rightarrow (48.0,\ 52.0)$

Sampling and estimation, in detail



CLT* 中心极限定理*: $\bar{X}$ near-normal

$$E(\bar{X}) = \mu,\ \mathrm{Var}(\bar{X}) = \frac{\sigma^2}{n}.$$

What makes a sample usable

a sample 样本	a small group chosen from the whole **population** 总体
a random sample 随机样本	needs **randomness** 随机性: every member has an equal chance of being chosen
why it matters	only then do the sample statistics estimate the population's without systematic error
biased 有偏 methods	sampling only volunteers, or the first 20 people to arrive – both over-represent a particular kind of person
the central limit theorem 中心极限定理	for a large n the sample mean is approximately normal, whatever the population's distribution
the consequence	$\bar{X} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$ – which is what every mean-based test rests on

> "Unsatisfactory" in a mark scheme means biased, and the mark is for naming *who* is over- or under-represented.

3 Hypothesis tests

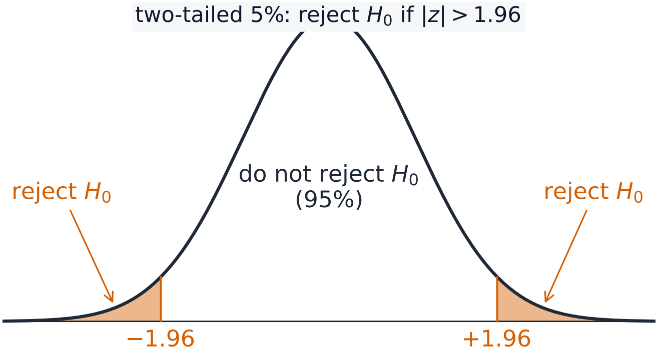
Hypothesis tests

Null 原假设 H_0 vs **alternative** 备择假设 H_1; pick a **significance level** 显著性水平.

> Test statistic in the **rejection region** 拒绝域 $\Rightarrow$ reject H_0.

Type I 第一类错误: reject a true H_0; **Type II** 第二类错误: accept a false H_0.

Hypothesis tests, in detail



> A two-tailed test at 5% rejects H_0 only if the test statistic falls in a shaded tail beyond ± 1.96.

The parts of a hypothesis test

a hypothesis test 假设检验	uses sample data to judge a claim
the null hypothesis 原假设	H_0 – the claim being tested, always an equality
the alternative 备择假设	H_1 – what you conclude if H_0 is rejected; one-tailed or two-tailed
the significance level 显著性水平	often 5% – fixed before you look at the data
the test statistic 检验统计量	worked out from the data, and compared with the critical region
the conclusion	reject or do not reject H_0 – in context, and never "accept H_0"

> Not rejecting is not proving. The test can only say the evidence was insufficient, which is why the wording of the conclusion is marked.

The parts of a test, and the two ways it fails

The hypotheses	H_0 states the value being tested; the **alternative hypothesis** 备择假设 H_1 states what you would accept instead, and its form decides one tail or two.
What is estimated	A sample gives **unbiased estimates** 无偏估计 of the mean and variance – and, for a proportion, of the **population proportion** 总体比例.
Type I error	A **type I error** 第一类错误: rejecting H_0 when it was true. Its probability *is* the significance level.
Type II error	A **type II error** 第二类错误: failing to reject H_0 when it was false. Lowering the significance level raises this one.

You never prove H_0. The conclusion is either "sufficient evidence to reject" or "insufficient evidence" – in the context of the question.

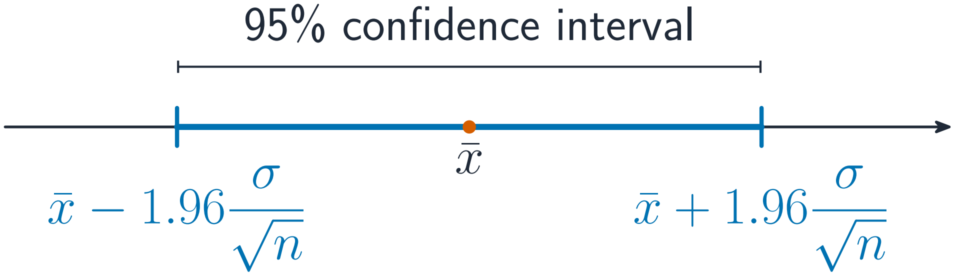
Worked example – a Poisson probability

$X \sim \mathrm{Po}(3)$. Find $P(X = 2)$.

$$P(X = 2) = e^{-3}\frac{3^2}{2!} = e^{-3} \times 4.5 = 0.224$$

For a Poisson variable the mean and variance both equal λ. Use $P(X = r) = e^{-\lambda}\frac{\lambda^r}{r!}$.

Sampling and estimation, continued



> A 95% confidence interval stretches 1.96 standard errors each side of the sample mean.

The Poisson distribution, continued

People arriving at random in a queue follow a Poisson distribution.

Exam tips

- Use the **Poisson** distribution for rare, random, independent events; its mean equals its variance $(= \lambda)$.
- When combining independent random variables, **variances add** (they never subtract).
- For a **hypothesis test**, state H_0 and H_1, the significance level, the test statistic, and a conclusion **in context**.
- For a **confidence interval**, use the correct z (or t) value and interpret it in words.

4

Recap

Topic 6 – key takeaways

1 Poisson

$\mathrm{Po}(\lambda)$; mean $=$ variance $= \lambda$

2 Combining

Var adds for independent variables

3 Sampling

CLT; $\bar{x} \pm 1.96\sigma/\sqrt{n}$

4 Testing

H_0 vs H_1; reject in the tail

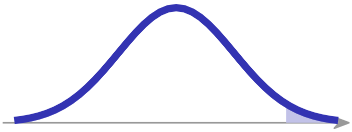
Check yourself

1. Mean and variance of $\mathrm{Po}(\lambda)$?

2. $\mathrm{Var}(2X)$ if $\mathrm{Var}(X) = 5$?

3. What does the area under $f(x)$ give?

4. Rejecting a true H_0 is which error?



Answers **1.** both λ **2.** $4 \times 5 = 20$ **3.** a probability **4.** Type I