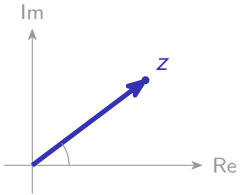
Pure Mathematics 3

A-Level Mathematics ▪ Topic 3

A-Level Mathematics



What we will cover

1. Partial fractions & integration
2. Vectors
3. Differential equations
4. Complex numbers
5. Recap

1 Algebra and integration

Partial fractions and integration

Split a proper rational function into simpler pieces, then integrate term by term:

$$\frac{1}{(x-1)(x+2)} = \frac{A}{x-1} + \frac{B}{x+2}.$$

Repeated factors need a squared denominator; a quadratic factor that will not factor stays as a linear numerator over it.

> $\int \frac{1}{x}\,dx = \ln|x| + C$. Partial fractions turn many rational integrals into logs.

Partial fractions: check the degree first

the precondition	the top must be of lower degree than the bottom
if it is not	"top-heavy": divide the polynomial first, giving a **quotient** 商 plus a proper remainder
distinct linear factors	$\dfrac{A}{ax+b} + \dfrac{B}{cx+d}$
a repeated factor	$\dfrac{A}{ax+b} + \dfrac{B}{(ax+b)^2}$
an irreducible quadratic	$\dfrac{Ax+B}{x^2+c}$ – a linear numerator over a quadratic
why bother	each piece integrates to a logarithm or a power, which the original fraction does not

> Skipping the degree check is the commonest error, and it produces an answer that cannot be right because the shapes do not match.

The binomial expansion for a rational power

The **binomial expansion 二项展开式** also works when the power is a fraction or is negative.

Logarithms, trigonometry and numerical methods (from Pure 2)

Subtopics 3.2, 3.3 and 3.6 are the same skills you met in Pure 2: the laws of logarithms with e^x and $\ln x$.

> The three new trig functions are the **reciprocals 倒数** of the familiar ones: the **secant 正割** $\sec\theta = \dfrac{1}{\cos\theta}$.

Differentiation

The methods are those of Pure 2 (the product, quotient and chain rules, with parametric and implicit curves).

One derivative is added – the **inverse tangent 反正切**:

$$\frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2}.$$

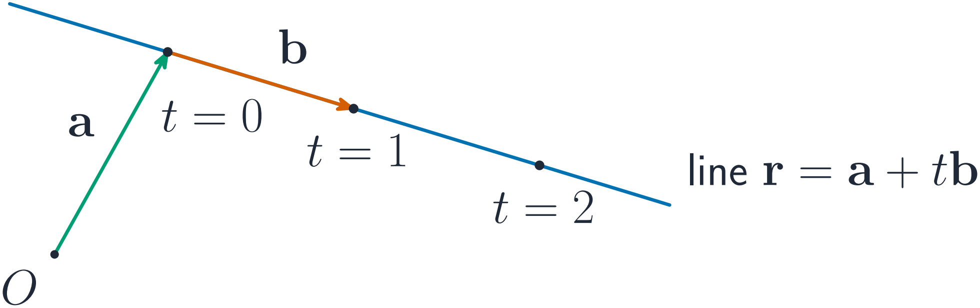
2 Vectors

Vectors

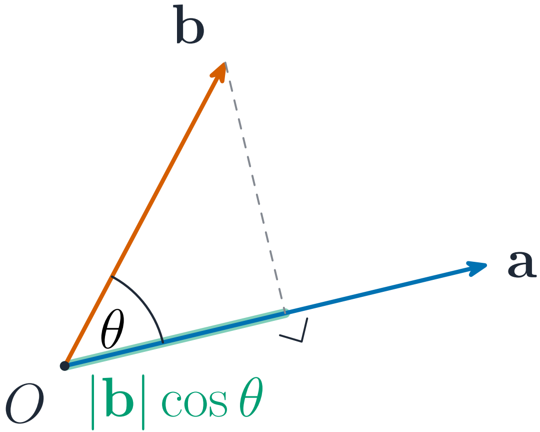
$|\mathbf{v}| = \sqrt{x^2 + y^2 + z^2}$; a **unit vector** 单位向量 has length 1.

$$\mathbf{r} = \mathbf{a} + t\mathbf{b} \text{ (a line).}$$

Lines can be parallel, intersect, or be **skew** 异面直线.



The scalar product



$\mathbf{a}\cdot\mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3 = |\mathbf{a}||\mathbf{b}|\cos\theta.$

> $\mathbf{a}\cdot\mathbf{b} = 0 \Rightarrow$ perpendicular.

e.g. angle between $(1,2,2)$ and $(2,2,1)$: $\cos\theta = \frac{8}{9}$, $\theta = 27.3°$.

3 Differential equations

Differential equations

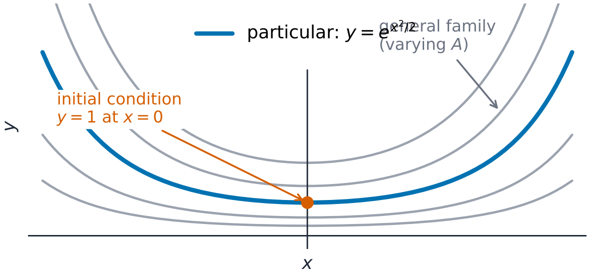
Separable 可分离变量: split y and x, then integrate both sides.

$\frac{dy}{dx} = xy$, $y = 1$ at $x = 0$

$$\ln y = \frac{1}{2}x^2 + c \Rightarrow y = e^{x^2/2}$$

A condition fixes C (the **particular solution** 特解).

Differential equations, in detail



general: family with A · IC → particular solution

$\frac{dy}{dx} = xy$ with $y = 1$ at $x = 0$ gives $\ln y = \frac{1}{2}x^2 + c$, so $y = e^{x^2/2}$. A condition fixes C (the **particular solution** 特解).

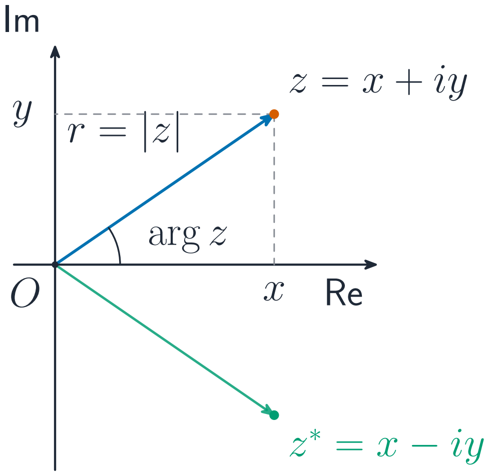
Separating the variables

a differential equation 微分方程	links a quantity to its rate of change
separable 可分离	when it can be written $\dfrac{dy}{dx} = f(x)g(y)$
the method	$\displaystyle\int \frac{dy}{g(y)} = \int f(x)\,dx$ – all the y's one side, all the x's the other
the general solution 通解	includes an arbitrary constant $+C$
the initial condition 初始条件	one given pair of values fixes C, giving the particular solution
the modelling step	read the sentence into an equation: "proportional to" means $= k\,(\ldots)$

Do not forget $+C$ before you apply the condition. Applying it afterwards to an equation with no constant leaves nothing to determine.

4 Complex numbers

Complex numbers



$z = x + iy$, $i^2 = -1$. **Conjugate** 共轭
$z^* = x - iy$.

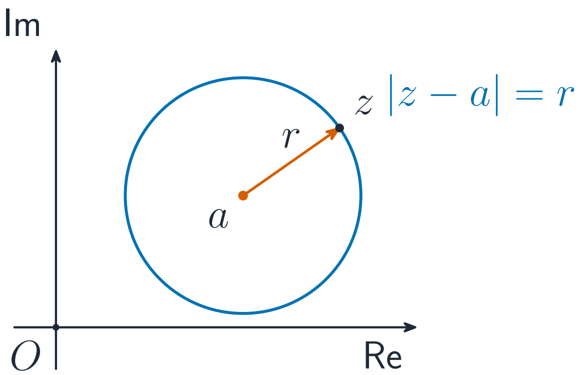
$$|z| = \sqrt{x^2 + y^2}, \quad z = re^{i\theta}.$$

Divide by multiplying by the conjugate:

$$\frac{3+i}{1-i}$$

$$= 1 + 2i$$

Loci on the Argand diagram



A condition on z draws a **locus** 轨迹:

- $|z-a|=r$ – a circle
- $|z-a|=|z-b|$ – a perpendicular bisector
- $\arg(z-a)=\theta$ – a half-line

Complex numbers: the five things to know

Cartesian form 直角坐标形式	$z = x + iy$; x is the real part, y the **imaginary part** 虚部
the conjugate 共轭	$z^* = x - iy$. For a polynomial with real coefficients, non-real roots come in conjugate pairs
the modulus 模	$\lvert z\rvert = \sqrt{x^2 + y^2}$ – its distance from the origin
the argument 辐角	the angle the point makes, measured from the positive real axis
polar form 极坐标形式	$z = r(\cos\theta + i\sin\theta) = re^{i\theta}$ – multiplying multiplies the moduli and adds the arguments
on the Argand diagram	$\lvert z\rvert$ is the distance from O, $\arg z$ the angle, and z^* the reflection in the real axis

Choose the form by the operation: Cartesian for adding, polar for multiplying, dividing and taking powers.

Worked example – integration by parts

Find $\int x \cos x \, dx$.

By parts with $u = x$, $\frac{dv}{dx} = \cos x$ (so $v = \sin x$):

$$\int x \cos x \, dx = x \sin x - \int \sin x \, dx$$

$$= x \sin x + \cos x + C$$

Choose u to be the part that gets simpler when differentiated – here $x \to 1$. The other part must be integrable.

Partial fractions

$$\frac{x+4}{(x+1)(x-2)} \xrightarrow{\text{split}} \frac{2}{x-2} - \frac{1}{x+1}$$

One fraction with a factorised bottom splits into a sum of simpler fractions.

Vectors, continued

Forces like wind and water are vectors – they have both size and direction.

The vectors vocabulary

position vector 位置向量	gives a point's place from the origin
displacement vector 位移向量	$\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$ – from one point to another
the line	$\mathbf{r} = \mathbf{a} + \lambda\mathbf{d}$: a point on it, plus a direction
the scalar product	$\mathbf{a} \cdot \mathbf{b} = \lvert\mathbf{a}\rvert\lvert\mathbf{b}\rvert \cos\theta$ – so it gives the angle
perpendicular	$\mathbf{a} \cdot \mathbf{b} = 0$
skew lines	in three dimensions two lines can be neither parallel nor intersecting – check both before concluding

"Do they meet?" is answered by solving for λ and μ in two components and testing the third. If it fails, the lines are skew.

Four more pieces of vocabulary

Integration by substitution	**Integration by substitution** 换元积分 reverses the chain rule – change the variable and the limits together.
Sign change and iteration	A **sign change** 符号变化 of a continuous f between a and b locates a root; an **iterative formula** 迭代公式 $x_{n+1} = g(x_n)$ then **converges** 收敛 to it when $\lvert g' \rvert < 1$ nearby.
Cosecant and cotangent	$\operatorname{cosec}\theta = 1/\sin\theta$ and $\cot\theta = 1/\tan\theta$ – each undefined wherever its denominator vanishes.
Magnitude	The **magnitude** 模 of a vector is $\lvert \mathbf{a} \rvert = \sqrt{a_1^2 + a_2^2 + a_3^2}$; dividing by it gives the unit vector.

State the interval, the two signs *and* the continuity. A sign change without continuity is an incomplete argument.

Exam tips

- Split a rational function into **partial fractions** before integrating or expanding.
- Choose the right **integration technique** (substitution, by parts, or partial fractions) from the form of the integrand.
- Use the scalar (dot) product for the **angle between vectors** and to test for perpendicularity; write a line as $\mathbf{r} = \mathbf{a} + t\mathbf{b}$.
- Give **complex numbers** in the form asked for (Cartesian or modulus-argument) and show them on an Argand diagram.

5 Recap

Topic 3 – key takeaways

1 Integration

partial fractions, by parts, substitution

2 Vectors

$\mathbf{r} = \mathbf{a} + t\mathbf{b}$; $\mathbf{a} \cdot \mathbf{b} = 0$ perpendicular

3 Diff. equations

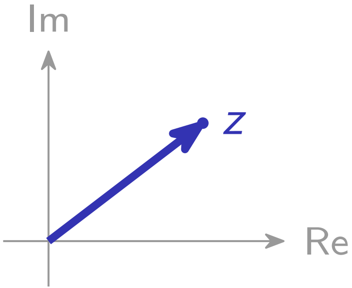
separate variables; a condition fixes C

4 Complex

$z = x + iy = re^{i\theta}$; loci on the Argand plane

Check yourself

1. What does $\mathbf{a} \cdot \mathbf{b} = 0$ tell you?
2. Which integration method suits $\int x \cos x \, dx$?
3. The conjugate of $3 - 2i$?
4. What locus is $|z - a| = r$?



Answers **1.** they are perpendicular **2.** by parts **3.** $3+2i$ **4.** a circle, centre a, radius r