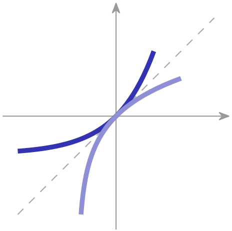
Pure Mathematics 2

A-Level Mathematics ▪ Topic 2

A-Level Mathematics



What we will cover

1. Modulus & polynomials
2. Logarithms & exponentials
3. Trigonometry
4. Differentiation
5. Integration & numerical methods
6. Recap

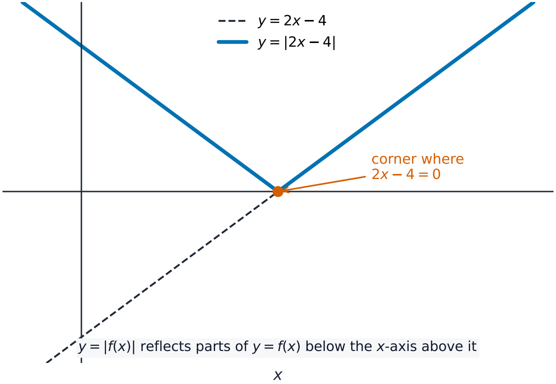
1 Algebra

Modulus and polynomials

$|x|$ is x if $x \geq 0$ and $-x$ if $x < 0$. The graph of $y = |f(x)|$ reflects anything below the x-axis above it.

> To solve $|f(x)| = k$, do $f(x) = k$ *and* $f(x) = -k$. Sketch first – a V-shape, vertex on the axis.

Modulus and polynomials, in detail



> Taking the modulus folds the part of the line below the axis upward into a V.

Worked example – a modulus inequality

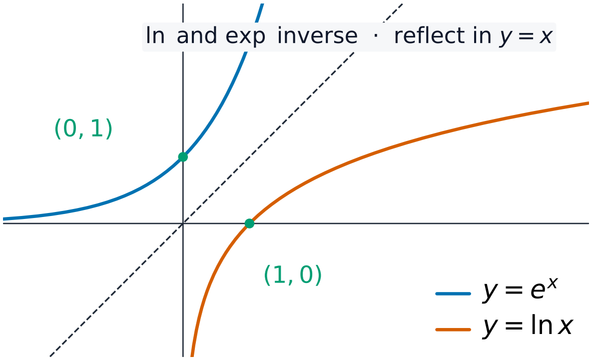
Solve $|3x+8| < 9$.

- $|A| < b$ means $-b < A < b$
- $-9 < 3x+8 < 9$
- $-17 < 3x < 1$
- $-\frac{17}{3} < x < \frac{1}{3}$

The double inequality is the safe method: it handles both branches at once, so no case can be forgotten. Squaring both sides also works, but only because both sides are non-negative here.

2 Logs and exponentials

Logarithms and exponentials



Laws of logs 对数定律:

$\log mn = \log m + \log n$,

$\log m^k = k \log m$.

> e^x and $\ln x$ are inverses. Unknown in the power $\Rightarrow$ take logs.

Linear form 线性形式:

$y = Ax^n \Rightarrow \ln y = \ln A + n \ln x$.

3 Trigonometry

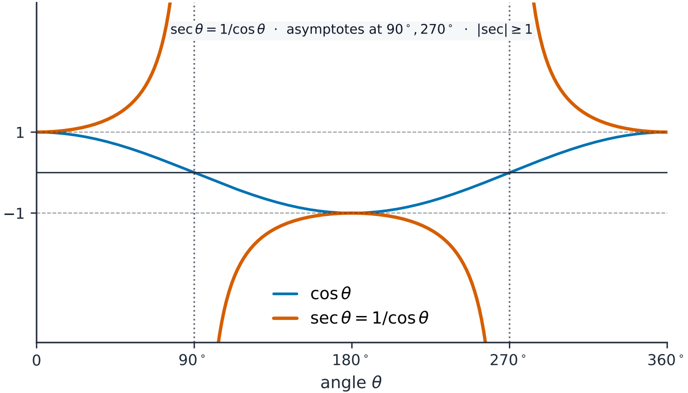
More trigonometry

Reciprocals: $\sec = \frac{1}{\cos}$, $\csc = \frac{1}{\sin}$, $\cot = \frac{1}{\tan}$.

$$\sec^2\theta \equiv 1 + \tan^2\theta.$$

Compound & **double angle** 二倍角; **R-formula** 辅助角公式
$a\sin\theta + b\cos\theta = R\sin(\theta + \alpha)$.

More trigonometry, in detail



Reciprocals: $\sec = \frac{1}{\cos}$, $\csc = \frac{1}{\sin}$, $\cot = \frac{1}{\tan}$.

Worked example – reduce to one function first

Solve $2\tan^2\theta + 3\sec\theta = 18$.

- replace $\tan^2\theta$ with $\sec^2\theta - 1$
- $2(\sec^2\theta - 1) + 3\sec\theta = 18$
- $2\sec^2\theta + 3\sec\theta - 20 = 0$
- a quadratic in $\sec\theta$: factorise, then take $\cos\theta = 1/\sec\theta$

Always reduce to one trig function before solving. The identity you need is the one that converts the squared term – here $\tan^2 \to \sec^2$.

The three reciprocal functions, and their identities

secant 正割	$\sec\theta = \dfrac{1}{\cos\theta}$
cosecant 余割	$\operatorname{cosec}\theta = \dfrac{1}{\sin\theta}$
cotangent 余切	$\cot\theta = \dfrac{1}{\tan\theta} = \dfrac{\cos\theta}{\sin\theta}$
from $\sin^2 + \cos^2 = 1$	divide by $\cos^2\theta$: $1 + \tan^2\theta \equiv \sec^2\theta$
and by $\sin^2\theta$	$1 + \cot^2\theta \equiv \operatorname{cosec}^2\theta$
compound angle 和角	$\sin(A \pm B)$, $\cos(A \pm B)$, $\tan(A \pm B)$ – and the double-angle forms that follow

"*co*" pairs with *sine*, not cosine: **cosecant** 余割 is $1/\sin$. Getting that backwards is the standard slip.

4 Calculus

Differentiation

Standard: $\frac{d}{dx}e^x = e^x$, $\frac{d}{dx}\ln x = \frac{1}{x}$, $\frac{d}{dx}\sin x = \cos x$.

Two rules

product 乘积法则: $(uv)' = u'v + uv'$

quotient 商法则: $\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$

Also **parametric** 参数方程 & **implicit** 隐式.

Parametric and implicit differentiation

parametric equations 参数方程	$x = x(t)$, $y = y(t)$ – both coordinates depend on a parameter
the gradient	$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$ – differentiate each with respect to t, then divide
implicitly 隐式 defined	an equation in x and y that is not solved for y
the method	differentiate every term with respect to x, using the chain rule on any y term: $\frac{d}{dx}(y^2) = 2y\frac{dy}{dx}$
then	collect the $\frac{dy}{dx}$ terms and make it the subject

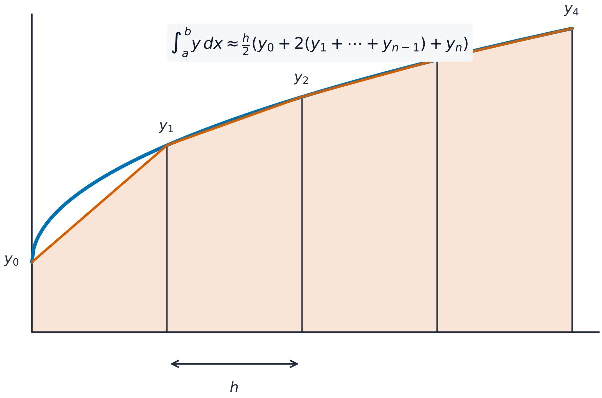
> In both cases you never solve for y first. That is the point of the techniques – they work exactly where solving is impossible.

Integration

Reverse each derivative, e.g. $\int e^{ax+b} dx = \frac{1}{a} e^{ax+b} + C$.

Trapezium rule 梯形法则 estimates an integral: $\frac{h}{2}[y_0 + y_n + 2(y_1 + \cdots + y_{n-1})]$.

Integration, in detail



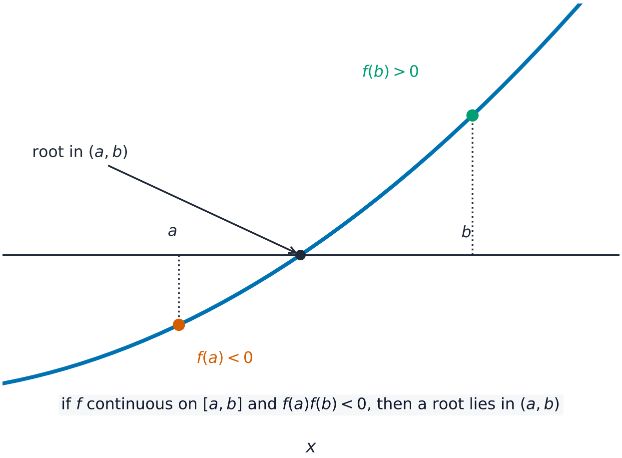
Reverse each derivative, e.g. $\int e^{ax+b}dx = \frac{1}{a}e^{ax+b} + C$.

Four techniques worth naming

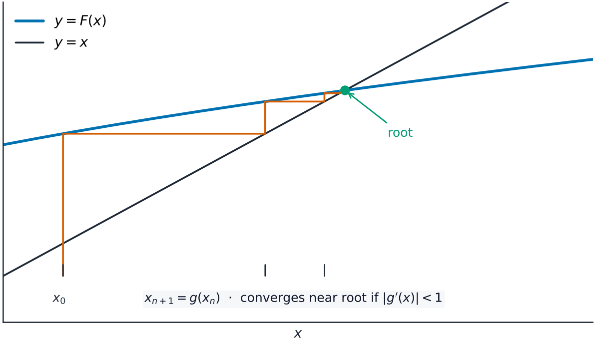
Product rule	The **product rule** 乘积法则: $(uv)' = u'v + uv'$ – for a product, never for a composition.
Integration by substitution	**Integration by substitution** 换元积分 reverses the chain rule: change the variable *and* the limits, or convert back at the end.
Natural logarithm	The **natural logarithm** 自然对数 is the inverse of e^x, and $\int \frac{1}{x}\,dx = \ln\lvert x\rvert + c$.
Remainder theorem	The **remainder theorem** 余数定理: the remainder when $f(x)$ is divided by $(x - a)$ is $f(a)$ – so a remainder of zero is the factor theorem.

Indices 指数 and **trigonometric identities** 三角恒等式 are what make an integral doable: simplify first, integrate second. A polynomial's **degree** 次数 is its highest power.

A sign change 变号 traps a root



Iteration 迭代: $x_{n+1} = F(x_n)$



iteration 迭代 $x_{n+1} = F(x_n)$

Finding a root when the equation cannot be solved

a root 根	a solution of $f(x) = 0$ – and many equations have no exact form for one
the sign change rule	if $f(a)$ and $f(b)$ have opposite signs and f is continuous, a root lies between them
iteration 迭代	rearrange to $x = F(x)$, then use the **iterative formula 迭代公式** $x_{n+1} = F(x_n)$
starting	begin from a value near the root, found by the sign change
converge 收敛	the sequence may converge to the root – or diverge, depending on the rearrangement chosen
the check	a sign change on a small interval either side confirms the root to the required accuracy

> Different rearrangements of the *same* equation converge at different speeds, or not at all. The exam gives you the rearrangement to use.

Worked example – the factor theorem

$p(x) = 2x^4 + kx^3 + kx^2 + 17x + 18$ has factor $(x+2)$. Find k.

Factor theorem: $p(-2) = 0$.

$2(16) - 8k + 4k - 34 + 18 = 16 - 4k = 0$

$\Rightarrow k = 4$

$(x+2)$ a factor $\Leftrightarrow p(-2) = 0$ – substitute the value that makes the bracket zero, not $+2$.

Logarithmic and exponential functions

Earthquake strength is measured on the logarithmic Richter scale.

Turning a power law into a straight line

the problem	$y = Ax^n$ is a curve, and its constants cannot be read off a plot
take logs	$\ln y = \ln A + n \ln x$
the linear form 线性形式	plotting $\ln y$ against $\ln x$ gives a straight line
reading it	the gradient is n and the intercept is $\ln A$
the other case	$y = Ab^x$ gives $\ln y = \ln A + x \ln b$: plot $\ln y$ against x, not against $\ln x$

Which variable to take logs of is the whole decision. Power law → log both; exponential → log y only.

Exam tips

- Use the **laws of logarithms** to solve equations; remember $\ln$ and e^x are inverses.
- For **numerical methods**, show a **sign change** to locate a root, set out the iteration clearly, and give the answer to the stated accuracy.
- Learn the **chain, product and quotient rules** and identify which the function needs.
- Solve modulus equations $|f(x)| = g(x)$ by considering both the positive and negative cases, and sketch to check.

5 Recap

Topic 2 – key takeaways

1 Algebra

factor theorem; modulus inequalities

2 Logs

e^x & $\ln x$ inverse; take logs for powers

3 Trig

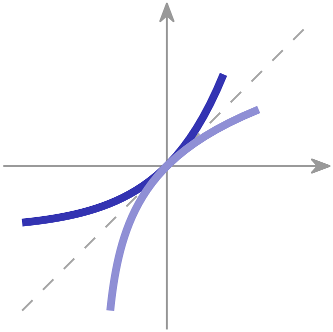
$\sec^2 = 1 + \tan^2$; R-formula

4 Calculus

product/quotient rules; trapezium rule

Check yourself

1. What is the remainder when $p(x) \div (x - a)$?
2. The derivative of $\ln x$?
3. Rewrite $\sec^2\theta$ using $\tan$.
4. What does a sign change between a and b show?



Answers **1.** $p(a)$ **2.** $1/x$ **3.** $1 + \tan^2\theta$ **4.** a root lies between them