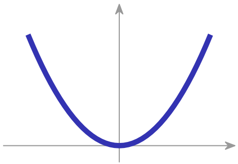
Pure Mathematics 1

A-Level Mathematics ▪ Topic 1

A-Level Mathematics



What we will cover

1. Quadratics & functions
2. Coordinate geometry
3. Circular measure & trigonometry
4. Series
5. Differentiation & integration
6. Recap

1 Quadratics and functions

Quadratics

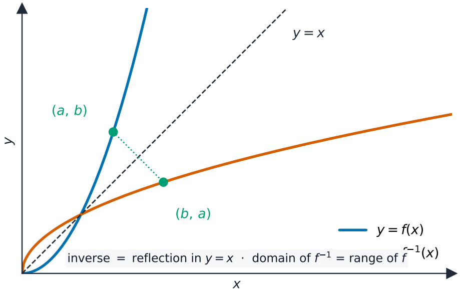
$ax^2 + bx + c = 0$ has discriminant $\Delta = b^2 - 4ac$.

$\Delta > 0$	$\Delta = 0$	$\Delta < 0$
two distinct real roots	one repeated real root	no real roots

Functions

- **domain** 定义域 (inputs), **range** 值域 (outputs)
- **one-one** 一一对应 $\Rightarrow$ has an **inverse** 反函数
- **composition** 复合函数 $fg(x) = f(g(x))$
- **transformations** 变换: $f(x) + a$, $f(x + a)$, $af(x)$, $f(ax)$

Functions, in detail



f^{-1} reflects f in $y = x$

Reflecting $y = f(x)$ in the line $y = x$ gives its inverse; a point (a, b) becomes (b, a).

The vocabulary of a function question

Composition of functions	$\mathrm{fg}(x)$ means do g first: the **composition of functions** 复合函数, and the order matters.
Increasing and decreasing	An **increasing function** 增函数 has $f'(x) > 0$ throughout an interval; a **decreasing function** 减函数 has $f'(x) < 0$.
Identity	An **identity** 恒等式 is true for *every* value of x – unlike an equation, which is true for some.
Coefficients	The **coefficients** 系数 are the numbers multiplying each power; a **quadratic inequality** 二次不等式 is solved from the sketch, not by dividing through.

> An inverse exists only for a one-one function – which is why a domain is so often restricted before an inverse is asked for.

Finding an inverse

To find f^{-1}: write $y = f(x)$, make x the subject, then swap the letters.

> The domain of f decides which square root you keep.

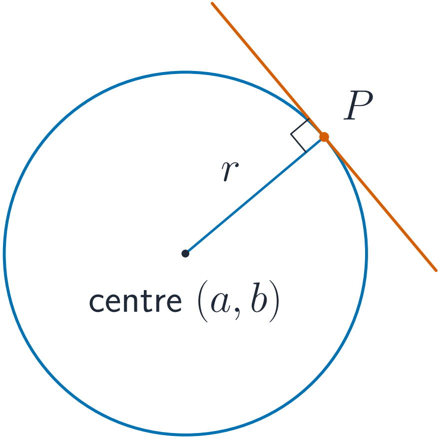
$f(x) = (x+3)^2 - 12$ for $x \geqslant 0$. Find f^{-1}.

$(x+3)^2 = y + 12 \quad \Rightarrow \quad x + 3 = \sqrt{y+12}$, taking the positive root because $x \geqslant 0$ means $x + 3 \geqslant 3 > 0$.

$$f^{-1}(x) = \sqrt{x+12} - 3.$$

2 Coordinate geometry

Coordinate geometry



Line gradient $m = \dfrac{y_2 - y_1}{x_2 - x_1}$;
perpendicular $\Rightarrow m_1 m_2 = -1$.

$$(x-a)^2 + (y-b)^2 = r^2.$$

> A **tangent** 切线 meets the **radius** 半径 at a right angle.

Straight lines

The **gradient** 斜率 through (x_1, y_1) and (x_2, y_2):

$$m = \frac{y_2 - y_1}{x_2 - x_1}.$$

Three forms of the same line:

$$y = mx + c, \quad y - y_1 = m(x - x_1), \quad ax + by + c = 0.$$

Parallel 平行: equal gradients.
Perpendicular 垂直: the gradients multiply to -1.

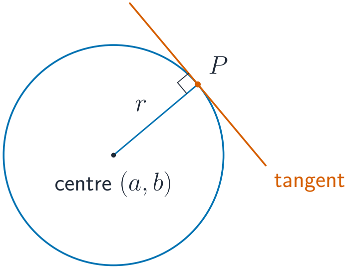
Circles

Centre (a, b), radius r:

$$(x-a)^2 + (y-b)^2 = r^2.$$

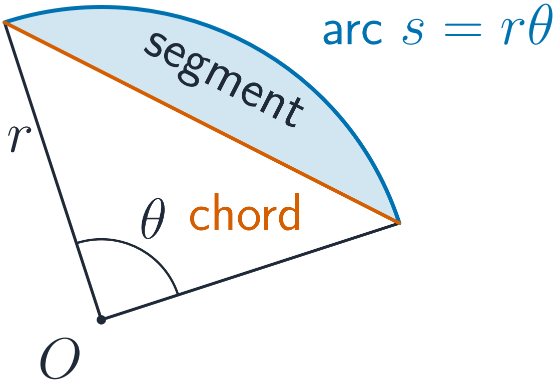
An expanded form such as $x^2 + y^2 - 6x + 10y - 27 = 0$ is the same circle – complete the square in x and in y to read off the centre and radius.

> A **tangent** 切线 meets the radius at a right angle. That one fact solves most circle problems.



3 Circular measure and trigonometry

Circular measure



Radians 弧度: $\pi = 180^\circ$. For a **sector** 扇形 (angle in radians):

$$s = r\theta, \quad A = \frac{1}{2}r^2\theta.$$

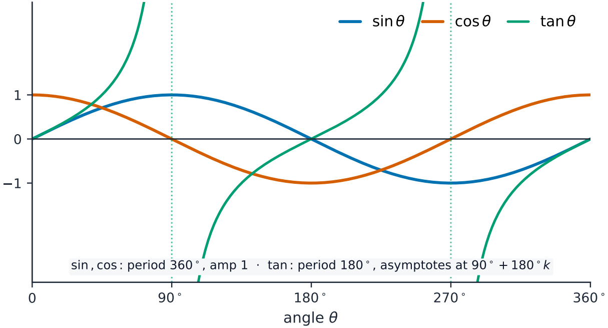
Segment 弓形 $= \frac{1}{2}r^2(\theta - \sin\theta)$.

Radians, circles, and the trigonometric words

Degrees and radians	$180^\circ = \pi$ radians. Every calculus formula for a trigonometric function assumes radians, never **degrees** 度.
Circle parts	The **diameter** 直径 is twice the radius; a **chord** 弦 joins two points on the circumference; arc length is $r\theta$ and sector area $\frac{1}{2}r^2\theta$, with θ in radians.
The functions	**Cosine** 余弦 and the **tangent function** 正切函数 join sine as the three ratios – and the tangent is undefined where the cosine is zero.
Principal value	An inverse trigonometric function returns only the **principal value** 主值; the other solutions come from the graph or the CAST diagram.

The **equation of a straight line** 直线方程 $y - y_1 = m(x - x_1)$ and $\tan\theta = m$ connect this topic to coordinate geometry.

Trigonometry



$\sin, \cos$ repeat every $360°$; learn exact values.

> **Identities** 恒等式: $\tan\theta \equiv \dfrac{\sin\theta}{\cos\theta}$, $\sin^2\theta + \cos^2\theta \equiv 1$.

Reduce to one function, then solve in the interval.

Solving trigonometric equations

Solve $6\sin\theta = 1 + \dfrac{2}{\sin\theta}$ **for** $-180° < \theta < 180°$**.**

Multiply by $\sin\theta$ – it is a quadratic in $\sin\theta$:

$$6\sin^2\theta - \sin\theta - 2 = 0 \Rightarrow (3\sin\theta - 2)(2\sin\theta + 1) = 0.$$

$\sin\theta = \frac{2}{3} \Rightarrow \theta = 41.8°,\ 138.2°;\quad \sin\theta = -\frac{1}{2} \Rightarrow \theta = -30°,\ -150°.$

Reduce to one function first, then find every solution in the interval – the marks are lost by stopping at the first one.

4 Series

Series

Binomial 二项展开式:

$$(a+b)^n = \sum \binom{n}{r} a^{n-r} b^r.$$

AP 等差数列: $u_n = a + (n-1)d.$
GP 等比数列: $u_n = ar^{n-1}.$

Sum to infinity ($|r| < 1$)

$$S_\infty = \frac{a}{1-r}.$$

e.g. $a = 32,\ r = -\frac{3}{4}$: $S_\infty = \frac{128}{7}$.

The binomial expansion

For a positive integer n,

$$(a+b)^n = a^n + \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \cdots + b^n,$$

with $\binom{n}{r} = \dfrac{n!}{r!\,(n-r)!}$.

First three terms of $(2-px)^5$

$$32 - 80px + 80p^2x^2 + \cdots$$

Keep the sign with the term: $(-px)^2$ is positive.

Arithmetic and geometric progressions

AP 等差数列 – adds a fixed **common difference** 公差 d:

$$u_n = a + (n-1)d, \quad S_n = \tfrac{n}{2}\big(2a + (n-1)d\big).$$

GP 等比数列 – multiplies by a fixed **common ratio** 公比 r:

$$u_n = ar^{n-1}, \quad S_n = \frac{a(1-r^n)}{1-r}.$$

The two progressions, and the binomial

Arithmetic progression

An **arithmetic progression** 等差数列 adds a common difference: $u_n = a + (n-1)d$, and $S_n = \frac{n}{2}(2a + (n-1)d)$.

Geometric progression

A geometric progression multiplies by a common ratio: $u_n = ar^{n-1}$, and the sum to infinity exists only when $|r| < 1$.

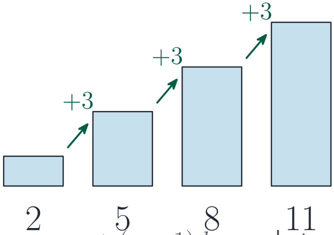
Binomial coefficient

The **binomial coefficient** 二项式系数 $\binom{n}{r}$ counts the ways of choosing r from n – and it is the coefficient in the expansion of $(a+b)^n$.

Check which progression before writing anything: a common *difference* or a common *ratio*, tested on the first three terms.

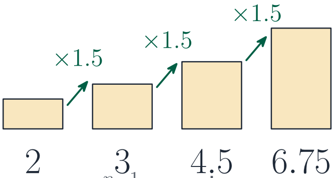
Arithmetic and geometric progressions, in detail

arithmetic: add d each step



$u_n = a + (n-1)d$: equal steps

geometric: multiply by r each step



$u_n = ar^{n-1}$: growing steps

An AP climbs in equal steps; a GP's steps grow by the same ratio

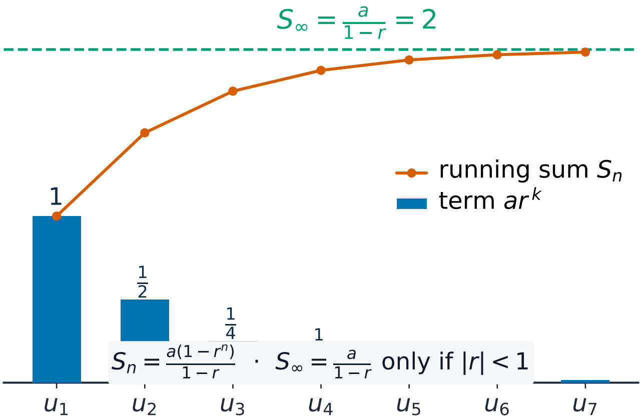
The sum to infinity

A GP **converges** 收斂 when $|r| < 1$, and then

$$S_\infty = \frac{a}{1-r}.$$

> $|r| < 1$ is a condition, not a formality – state it before you use the formula.

The sum to infinity, in detail



The bars shrink to zero while the running sum approaches $a/(1-r)$

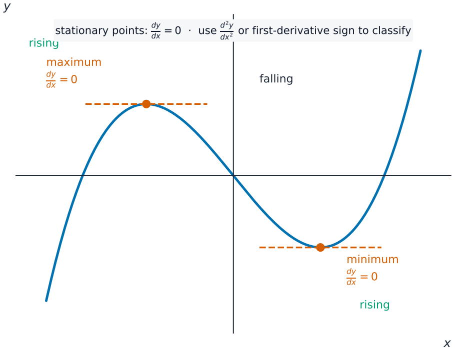
5 Calculus

Differentiation

$$\frac{d}{dx}x^n = nx^{n-1}; \quad \text{chain: } f'(g)\,g'.$$

Stationary point 驻点: $\frac{dy}{dx} = 0$. Then $f'' > 0$ min, $f'' < 0$ max.

Differentiation, in detail



At a stationary point the tangent is horizontal. The second derivative then tells you which: $f'' > 0$ is a minimum, $f'' < 0$ a maximum.

The rules of differentiation

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

Chain rule. $\dfrac{dy}{dx} = \dfrac{dy}{du} \cdot \dfrac{du}{dx}$ **Product rule.** $(uv)' = u'v + uv'$ **Quotient rule.** $\left(\dfrac{u}{v}\right)' = \dfrac{u'v - uv'}{v^2}$

> Read the function's shape first: a power, a function of a function, a product, or a quotient. The shape names the rule.

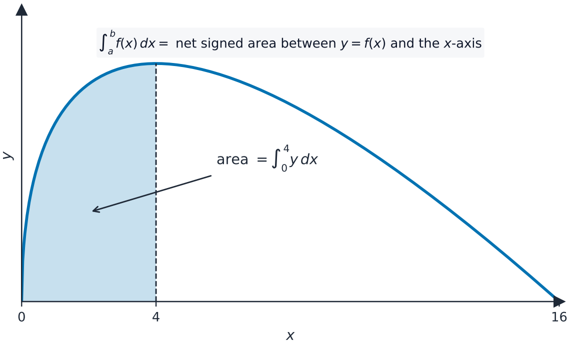
Integration

Reverse of **differentiation** 微分 $(+C)$:

$$\int (ax+b)^n dx = \frac{(ax+b)^{n+1}}{a(n+1)} + C.$$

Area $= \displaystyle\int_p^q y\,dx.$

Integration, in detail



A definite integral is the signed area between the curve and the x-axis – area below the axis counts as negative.

Definite integrals and area

$$\int_a^b f(x)\,dx = \Big[F(x)\Big]_a^b = F(b) - F(a),$$

the area between the curve and the x-axis from $x = a$ to $x = b$. Between two curves, integrate the difference – upper minus lower.

> Area below the axis integrates to a negative number. If a question asks for an area that crosses the axis, split the integral at the crossing.

Using the derivative, and improper integrals

a rate of change 变化率	a derivative *is* one – and linked rates use the chain rule: $\dfrac{dV}{dt} = \dfrac{dV}{dr} \times \dfrac{dr}{dt}$
stationary points	where $\dfrac{dy}{dx} = 0$; the second derivative's sign classifies them
increasing / decreasing	where the derivative is positive / negative
tangent and normal	gradients m and $-\dfrac{1}{m}$ at the same point
an improper integral 广义积分	has an infinite limit, or an endpoint where the integrand is undefined
how to handle it	replace the offending limit with a variable, integrate, then take the limit

For a linked rate, write down which quantity you know the rate of and which you want, then chain the derivatives between them. That sentence is most of the mark.

What the derivative tells you about the shape

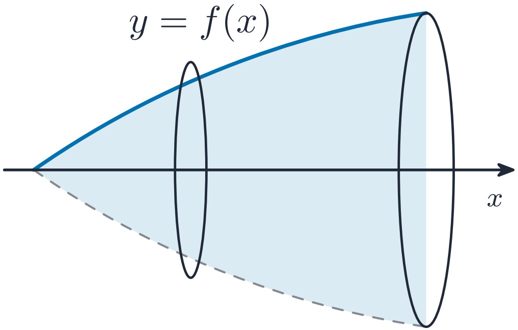
Gradient of a curve	$f'(x)$ is the **gradient of a curve** 曲线的梯度 at a point – the gradient of the tangent there, as a **chord** 弦 shortens to nothing.
Maximum and minimum	At a **maximum point** 极大值点 $f' = 0$ and $f'' < 0$; at a **minimum point** 极小值点 $f' = 0$ and $f'' > 0$.
Point of inflexion	A **point of inflexion** 拐点 is where the curvature changes sign. A stationary one has $f' = 0$ and $f'' = 0$ – but $f'' = 0$ alone is not enough.
Integration	Reversing the process needs the **constant of integration** 积分常数: one condition fixes it, and without it the answer is a family of curves.

Test a stationary point with the second derivative *or* with the sign of f' either side. Both are accepted; neither may be skipped.

Volume of revolution

Rotate a **region** 区域 about the x-axis – each slice is a disc of area πy^2:

$$V = \pi \int_p^q y^2 \, dx.$$



Worked example – completing the square

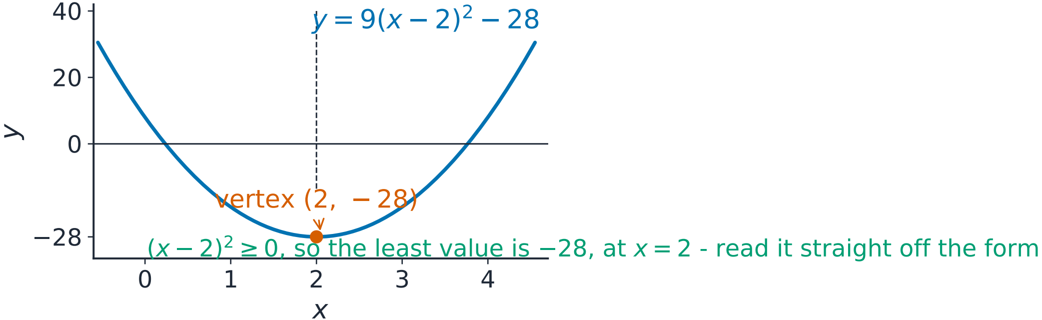
Write $9x^2 - 36x + 8$ as $p(x+q)^2 + r$.

Take out 9 from the first two terms, complete the square inside:

$$9(x^2 - 4x) + 8 = 9\big((x-2)^2 - 4\big) + 8$$
$$= 9(x-2)^2 - 28$$

Factor out the x^2 coefficient before completing the square. The form gives the minimum instantly: $(2, -28)$.

The completed square hands you the vertex



$y = a(x-h)^2 + k$ has vertex (h, k) · min if $a > 0$, max if $a < 0$

> The completed square hands you the vertex: least value -28 at x equals 2

Worked example – a maximum by differentiation

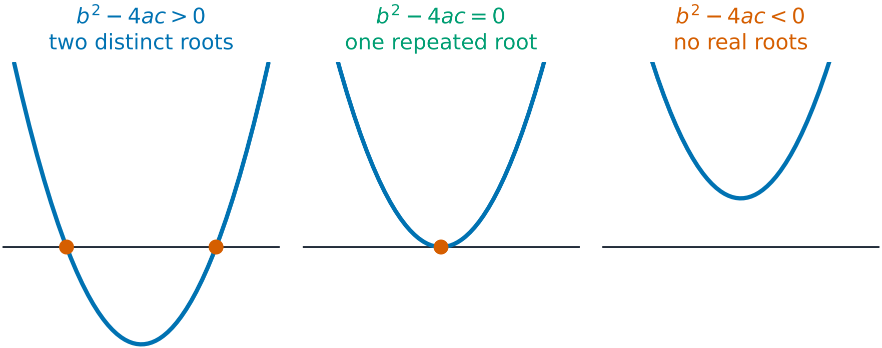
$y = 4x^{1/2} - x$ has a maximum at $x = a$. Find a.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 2x^{-1/2} - 1 = 0 \Rightarrow x^{1/2} = 2$$

so $a = 4$ (and $y = 4 \cdot 2 - 4 = 4$)

Rewrite roots as powers ($\sqrt{x} = x^{1/2}$) before differentiating. Set $\mathrm{d}y/\mathrm{d}x = 0$ for a turning point.

The discriminant



$$ax^2 + bx + c = 0 \;\Rightarrow\; x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad \text{(real roots need } b^2 - 4ac \geq 0\text{)}$$

> The sign of $b^2 - 4ac$ decides how many times the parabola meets the x-axis.

Solving quadratic equations and inequalities

Three ways to solve $ax^2 + bx + c = 0$:
factorise, complete the square, or

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

For an **inequality** 二次不等式, find the roots first, then decide which side of each root makes the statement true.

> Sketch the **parabola** 抛物线: it is above the x-axis *outside* the roots and below it *between* them. So $(k-4)(k-16) > 0$ gives $k < 4$ or $k > 16$.

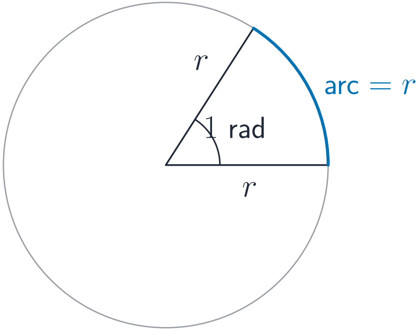
Simultaneous equations, and quadratics in disguise

One linear, one quadratic. Rearrange the linear one for a letter, substitute into the quadratic, and you are left with a single quadratic to solve.

Quadratic in disguise. $x^4 - 5x^2 + 4 = 0$ is quadratic in x^2: let $u = x^2$, solve $u^2 - 5u + 4 = 0$, then go back to x.

> The same trick returns in trigonometry, with $\sin\theta$ in place of u.

Radians



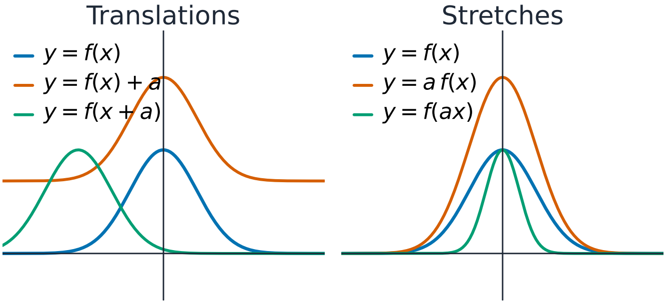
one **radian**: the angle whose arc is exactly one radius long

π rad $= 180°$, so 1 rad $\approx 57.3°$

degrees $\rightarrow$ radians: $\times \frac{\pi}{180}$ radians $\rightarrow$ degrees: $\times \frac{180}{\pi}$

One radian: the angle whose arc equals the radius

Graphs of inverses and transformations



$f(x) + a$ moves up · $f(x + a)$ moves left · $af(x)$ stretch y · $f(ax)$ compress x if $a > 1$

> Adding to the output or input slides the curve; a multiplier stretches it.

The transformations, and stating them fully

$y = f(x) + a$	translation up by a
$y = f(x + a)$	translation left by a – the opposite of the sign
$y = af(x)$	stretch parallel to the y-axis, scale factor a
$y = f(ax)$	stretch parallel to the x-axis, scale factor $\frac{1}{a}$
$y = -f(x)$ **and** $y = f(-x)$	reflection in the x-axis, and in the y-axis
combining two	the order can matter – so state each one fully: type, direction and magnitude

An inverse function's graph is the reflection of f in the line $y = x$, which is why a function must be one-to-one before an inverse exists.

Trigonometry, continued

> A Ferris wheel: a point on the rim rises and falls like a sine curve.

The exact values, and the progressions

sin	$\sin 30^\circ = \frac{1}{2}$, $\sin 45^\circ = \frac{1}{\sqrt{2}}$, $\sin 60^\circ = \frac{\sqrt{3}}{2}$
cos	$\cos 30^\circ = \frac{\sqrt{3}}{2}$, $\cos 45^\circ = \frac{1}{\sqrt{2}}$, $\cos 60^\circ = \frac{1}{2}$
tan	$\tan 30^\circ = \frac{1}{\sqrt{3}}$, $\tan 45^\circ = 1$, $\tan 60^\circ = \sqrt{3}$
a progression 数列	a list of terms following a rule
arithmetic 等差数列	a common difference d: $u_n = a + (n-1)d$, and $S_n = \frac{n}{2}\left(2a + (n-1)d\right)$
geometric 等比数列	a common ratio r: $u_n = ar^{n-1}$, and $S_\infty = \dfrac{a}{1-r}$ when $\lvert r \rvert < 1$

> The exact values come from the half-equilateral triangle and the half-square. Reconstructing them from those two triangles beats memorising twelve fractions.

Exam tips

- Show **every line of algebra** – method marks are lost by jumping steps; use the **discriminant** $b^2 - 4ac$ to decide the number of real roots.
- Work in **radians** for circular measure (arc $= r\theta$, sector area $= \frac{1}{2}r^2\theta$) and for calculus of trig functions.
- For **differentiation**, set $\frac{dy}{dx} = 0$ for stationary points and use the second derivative to classify them.
- For **integration**, add $+c$ to an indefinite integral and use limits for area; an area below the axis gives a negative integral.

6 Recap

Topic 1 – key takeaways

1 **Quadratics**

discriminant decides the roots

2 **Geometry**

$m_1 m_2 = -1$; tangent $\perp$ radius

3 **Trig**

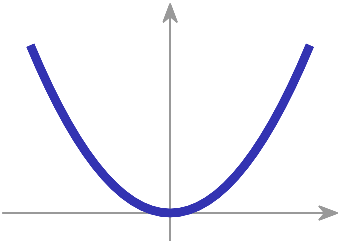
$\sin^2 + \cos^2 = 1$; sectors in radians

4 **Calculus**

$\frac{dy}{dx} = 0$ stationary; $\int y\,dx$ area

Check yourself

1. How many roots if $b^2 - 4ac < 0$?
2. Reflect a graph in which line to get its inverse?
3. Arc length of a sector (radians)?
4. Test for a minimum stationary point?



Answers **1.** none **2.** $y = x$ **3.** $s = r\theta$ **4.** $f''(x) > 0$