Week 21

A-Level Mathematics

Homework bundle for this week

本周作业合集

exercises.lol

Contents 目录

A-Level Mathematics Name: ____________ Score: ______

1. Writing (x+4)/((x+1)(x−2)) = A/(x+1) + B/(x−2), substitute x = 2 to find B.

   ____

2. Using $\sec^2\theta \equiv 1 + \tan^2\theta$, if $\tan\theta = 3$, what is $\sec^2\theta$?

   ____

3. The derivative of $\tan^{-1}x$ is $1/(1+x^2)$. What is its value at x = 1?

   ____

4. What is $\int 2x/(x^2+1)\,dx$?

   - ☐ $\ln(x^2+1) + C$
   - ☐ $2\ln(x) + C$
   - ☐ $x^2+1 + C$
   - ☐ $\tan^{-1}x + C$

5. What is the magnitude of the vector i + 2j + 2k?

   ____

6. To solve a separable first-order differential equation, you:

   - ☐ put all y terms on one side and all x terms on the other, then integrate both sides
   - ☐ differentiate both sides
   - ☐ square both sides
   - ☐ use partial fractions only

7. What is the modulus of the complex number 3 + 4i?

   ____

8. The binomial expansion $(1+x)^n$ for a fractional or negative n is valid only when:

   - ☐ $|x| < 1$
   - ☐ $x > 1$
   - ☐ $x = 0$
   - ☐ n is positive

9. The integration-by-parts formula is $\int u\,dv$ =:

   - ☐ $uv - \int v\,du$

- ☐ $uv + \int v\, du$
- ☐ $u\, du - v\, dv$
- ☐ $\int u\, du \times \int v\, dv$

10. An initial condition is used to:
    - ☐ find the constant, giving the particular solution
    - ☐ remove the derivative
    - ☐ make the equation linear
    - ☐ check the units

A-Level Mathematics

1. **2**

   At $x = 2$: $2 + 4 = B(2 + 1)$, so $6 = 3B$, giving $B = 2$.

2. **10**

   $\sec^2\theta = 1 + 3^2 = 1 + 9 = 10$.

3. **0.5**

   $1/(1 + 1^2) = 1/2 = 0.5$.

4. **$\ln(x^2+1) + C$**

   The top is the derivative of the bottom ($k\, f'/f$ form), so the integral is $\ln(x^2+1) + C$.

5. **3**

   $|v| = \sqrt{(1^2 + 2^2 + 2^2)} = \sqrt{(1+4+4)} = \sqrt{9} = 3$.

6. **put all y terms on one side and all x terms on the other, then integrate both sides**

   Separate the variables, then integrate each side.

7. **5**

   $|z| = \sqrt{(3^2 + 4^2)} = \sqrt{(9 + 16)} = \sqrt{25} = 5$.

8. **$|x| < 1$**

   For non-positive-integer powers the series converges only for $|x| < 1$.

9. **$uv - \int v\, du$**

   Integration by parts: $\int u\, dv = uv - \int v\, du$.

10. **find the constant, giving the particular solution**

    A known value fixes the arbitrary constant, turning the general solution into the particular one.

3.1 Algebra (partial fractions, rational-power binomial)

Name: ______________ Class: __________ Date: ____________ **Total: 17 marks**

KEY WORDS partial fractions 部分分式 • binomial expansion 二项展开式 • further algebra 代数 • rational function
• converges 收敛

Objective

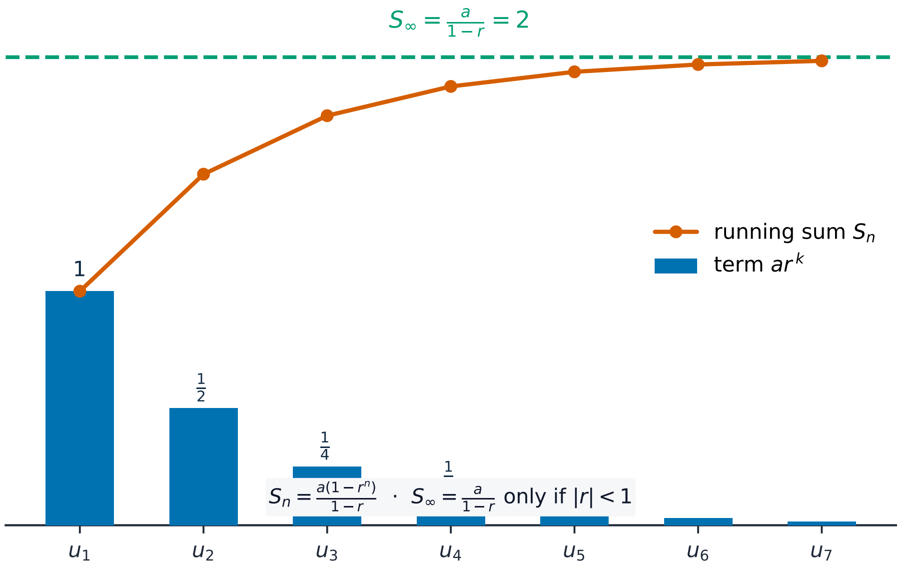
Build the skills to answer exam questions on **further algebra 代数** — **partial fractions 部分分式** and the **binomial expansion for a rational power 分数次幂二项展开式**.

You must be able to:

- split a rational function into partial fractions (including a repeated factor)
- expand $(1+x)^n$ for fractional/negative n (for $|x|<1$)
- combine the two (expand after splitting)

1 Worked examples

■ **Partial fractions & rational-power binomial**



The rational-power binomial series converges only for $|x| < 1$ — the terms must shrink

$\frac{x+4}{(x+1)(x-2)} = \frac{A}{x+1} + \frac{B}{x-2}$: $x + 4 = A(x-2) + B(x+1)$; $x = 2 \Rightarrow B = 2$; $x = -1 \Rightarrow A = -1$. So $\frac{2}{x-2} - \frac{1}{x+1}$.

$(1+x)^{1/2} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \cdots$ for $|x| < 1$.

2 Practice

2.1 Find the first three terms of $(1+x)^{-1}$. [2]

2.2 Write $\frac{5}{(x-1)(x+4)}$ in the form $\frac{A}{x-1} + \frac{B}{x+4}$. [3]

3 Exam-style questions

3.1 The expansion of $(1+2x)^{1/2}$ in ascending powers of x begins: [1]

- **A** $1+x-\frac{1}{2}x^2+\cdots$
- **B** $1+x+\frac{1}{2}x^2+\cdots$
- **C** $1+2x-2x^2+\cdots$
- **D** $1-x+x^2+\cdots$

3.2 Express $\dfrac{9x+5}{(x+1)(2x-1)}$ in partial fractions. [4]

3.3 (a) Express $\dfrac{2}{(1-x)(1+2x)}$ in partial fractions. [3]

(b) Hence find the expansion of $\dfrac{2}{(1-x)(1+2x)}$ up to the term in x^2. [4]

Solutions

2.1 $(1+x)^{-1} = 1 - x + x^2 - \cdots.$

2.2 $5 = A(x+4) + B(x-1)$; $x = 1 \Rightarrow A = 1$; $x = -4 \Rightarrow B = -1$; $\dfrac{1}{x-1} - \dfrac{1}{x+4}$.

3.1 **A**. $(1+2x)^{1/2} = 1 + \frac{1}{2}(2x) + \frac{(1/2)(-1/2)}{2}(2x)^2 + \cdots = 1 + x - \frac{1}{2}x^2 + \cdots.$

3.2 $9x + 5 = A(2x-1) + B(x+1)$; $x = -1 \Rightarrow -4 = -3A \Rightarrow A = \frac{4}{3}$; $x = \frac{1}{2} \Rightarrow 9.5 = 1.5B \Rightarrow B = \frac{19}{3}$; $\dfrac{4}{3(x+1)} + \dfrac{19}{3(2x-1)}$.

3.3 (a) $2 = A(1+2x) + B(1-x)$; $x = 1 \Rightarrow A = \frac{2}{3}$; $x = -\frac{1}{2} \Rightarrow B = \frac{4}{3}$; $\dfrac{2/3}{1-x} + \dfrac{4/3}{1+2x}$.
(b) $\frac{2}{3}(1 + x + x^2) + \frac{4}{3}(1 - 2x + 4x^2)$; $= \frac{2}{3} + \frac{4}{3} + (\frac{2}{3} - \frac{8}{3})x + (\frac{2}{3} + \frac{16}{3})x^2$; $= 2 - 2x + 6x^2$.

10 Let $f(x) = \dfrac{x^2 + 2x - 11}{(3+x)\left(2+x^2\right)}$.

(a) Express $f(x)$ in partial fractions. [6]

(b) Hence obtain the expansion of $f(x)$ in ascending powers of x, up to and including the term in x^2. [5]

1 **(a)** Sketch the graph of $y = |2x - 3|$. [1]

(b) Solve the inequality $3x - 1 < |2x - 3|$. [2]

5 The polynomial $3x^3 + pax^2 + 7a^2x + qa^3$ is denoted by $f(x)$, where p, q and a are constants and $a \neq 0$.

When $f(x)$ is divided by $(x + 2a)$ the remainder is $-22a^3$. When $f(x)$ is divided by $(3x - a)$ the remainder is $-a^3$.

Find the values of p and q. [5]

10 **(a)** Find the quotient and remainder when $x^3 + 5x^2 - 2x - 15$ is divided by $x^2 - 3$. [3]

(b) The variables x and y satisfy the differential equation

$$\frac{dy}{dx} = \frac{x^3 + 5x^2 - 2x - 15}{6y(x^2 - 3)}.$$

It is given that $y = 2$ when $x = 2$.

Solve the differential equation to obtain an expression for y^2 in terms of x. [5]

2 **(a)** Expand $(6-x)(1-2x)^{-\frac{3}{2}}$ in ascending powers of x, up to and including the term in x^2, simplifying the coefficients. [4]

(b) State the set of values of x for which the expansion is valid. [1]

10 **(a)** Find the quotient and remainder when x^2 is divided by $1+4x^2$. [2]

(b) Find the exact value of $\int_0^{0.5} x\tan^{-1}(2x)\,dx$. [6]

3.2 Logarithmic and exponential functions

Name: ______________ Class: _________ Date: ____________ **Total: 13 marks**

KEY WORDS logarithmic and exponential functions 对数与指数函数

Objective

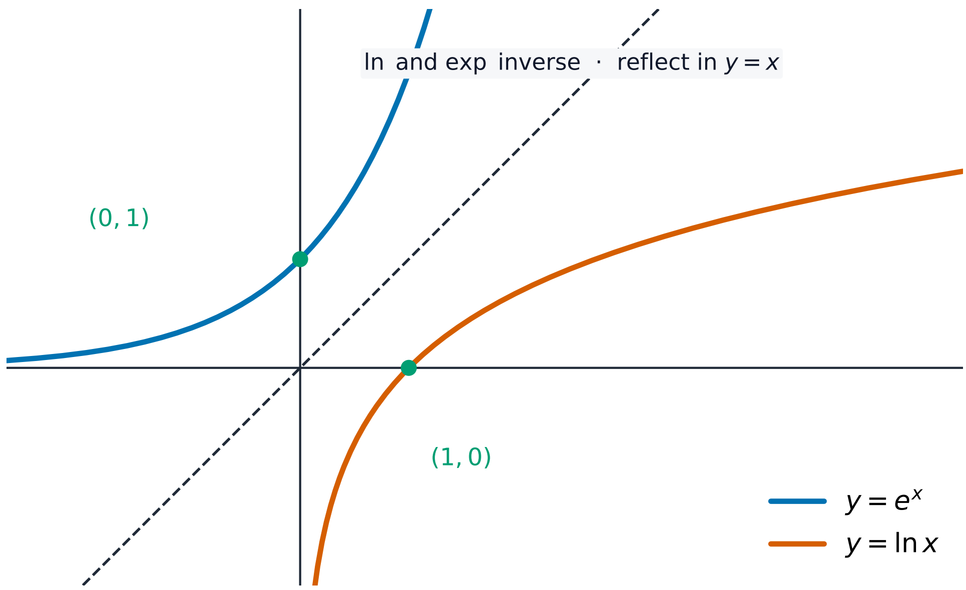
Build the skills to answer exam questions on **logarithmic and exponential functions 对数与指数函数** in Pure 3 — the same log/exp toolkit as Pure 2, applied to equations and modelling.

You must be able to:

- use the laws of logarithms and $e^x/\ln x$ as inverses
- solve equations with the unknown in the power or inside a log
- handle exponential models (growth/decay)

1 Worked examples

Solving log/exp equations



e^x and $\ln x$ undo each other — take logs to free an unknown power

Solve $e^{2x} - 5e^x + 6 = 0$**.** Let $u = e^x$: $u^2 - 5u + 6 = 0 \Rightarrow (u-2)(u-3) = 0$. So $e^x = 2$

or 3, giving $x = \ln 2$ or $\ln 3$.

2 Practice

2.1 Solve $\ln(x-1) = 2$, giving x to 3 s.f. [2]

2.2 Write $2\ln a - \ln b$ as a single logarithm. [1]

3 Exam-style questions

3.1 The solution of $5e^{-x} = 2$ is: [1]

- **A** $\ln 2.5$
- **B** $-\ln 2.5$
- **C** $\ln 0.4$
- **D** $\ln 10$

3.2 Solve the equation $e^{2x} - 7e^{x} + 10 = 0$, giving exact answers. [4]

3.3 A quantity decays so that $P = P_0 e^{-kt}$. After 5 years P has fallen to half of P_0.

(a) Find k, to 3 s.f. [3]

(b) Find the time for P to fall to one tenth of P_0. [2]

Solutions

2.1 $x - 1 = e^2 \Rightarrow x = 1 + e^2 = 8.39$.

2.2 $\ln \dfrac{a^2}{b}$.

3.1 **A**. $e^{-x} = 0.4 \Rightarrow -x = \ln 0.4 \Rightarrow x = -\ln 0.4 = \ln 2.5$.

3.2 let $u = e^x$: $u^2 - 7u + 10 = 0 \Rightarrow (u-2)(u-5) = 0$; $e^x = 2$ or 5; $x = \ln 2$ or $\ln 5$.

3.3 (a) $\frac{1}{2}P_0 = P_0 e^{-5k} \Rightarrow e^{-5k} = 0.5 \Rightarrow -5k = \ln 0.5$; $k = 0.139$.
(b) $0.1 = e^{-kt} \Rightarrow t = \dfrac{\ln 0.1}{-0.139} = 16.6$ years.

2 **(a)** Show that the equation $\log_4(2x+1) = 2\log_4(3x-1) - 2$ can be written as a quadratic equation in x. [3]

(b) Hence solve the equation $\log_4(2x+1) = 2\log_4(3x-1) - 2$. [2]

2 It is given that $2\ln p + \ln(p-1) - \frac{1}{2}\ln(q+1) = 3$.

Find q in terms of p. [3]

1 Solve the equation $\dfrac{\mathrm{e}^{\text{[illegible]}} + 2\mathrm{e}^{\text{[illegible]}}}{\mathrm{e}^{x} - 3} = 4$. Give your answer correct to 3 decimal places. [5]

A-LEVEL MATHEMATICS • EXERCISE SHEET

3.3 Trigonometry (identities, R-formula)

Name: ________ Class: ________ Date: ________ **Total: 13 marks**

KEY WORDS r-formula 辅助角公式 • trigonometry 三角学

Objective

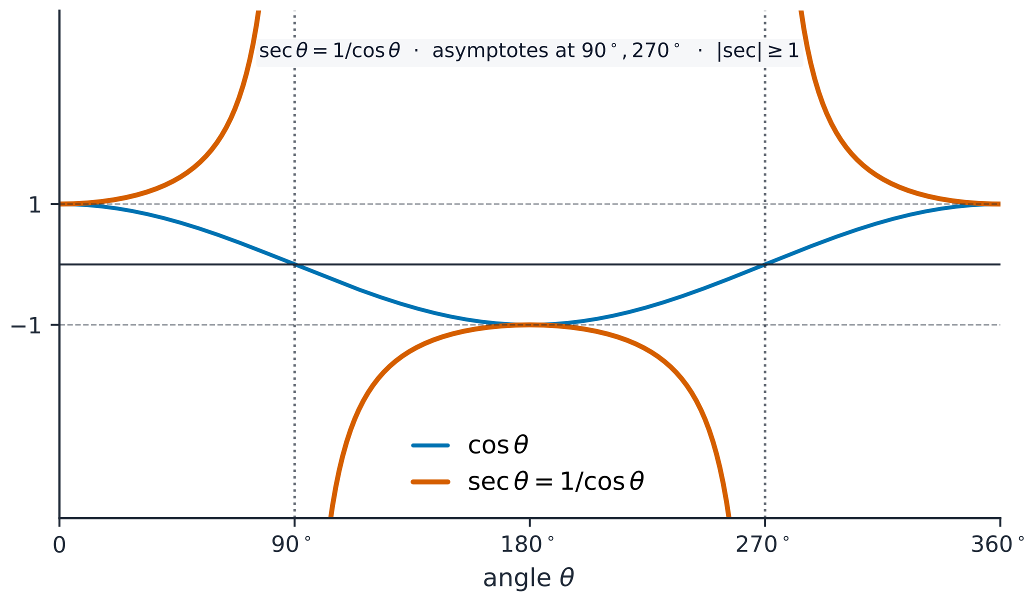
Build the skills to answer exam questions on **trigonometry** 三角学 in Pure 3 — the reciprocal identities, compound- and double-angle formulae, and the **R-formula** 辅助角公式, used in proofs and equations.

You must be able to:

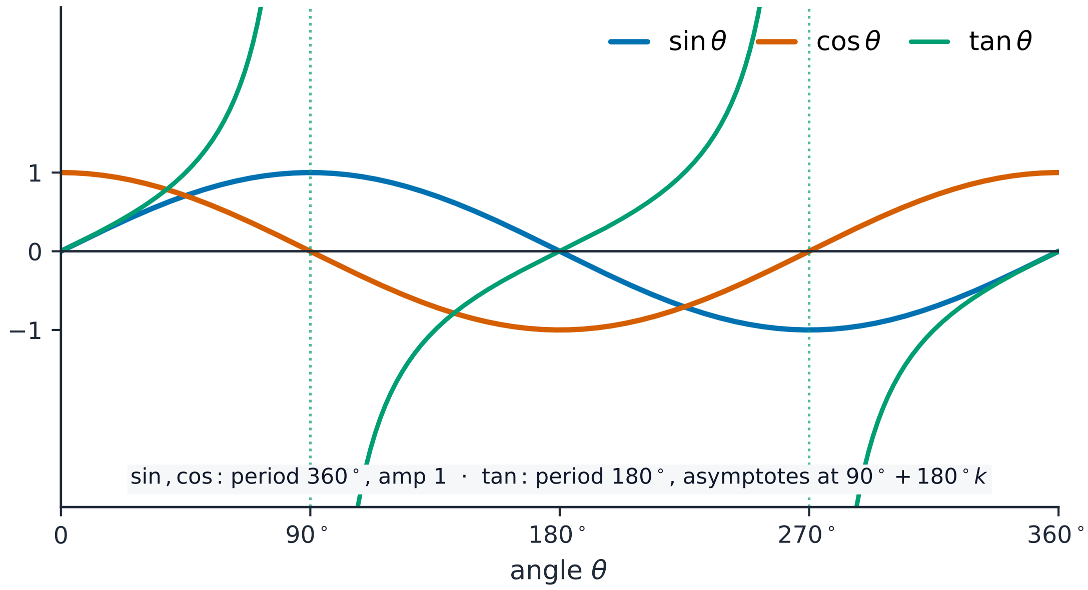
- prove identities using $\sec^2\theta \equiv 1 + \tan^2\theta$ etc.
- expand and use compound/double-angle formulae
- express $a\sin\theta + b\cos\theta$ as $R\sin(\theta \pm \alpha)$ and solve

1 Worked examples

■ R-formula in an equation



The reciprocal functions and the identities let you reduce an equation to one function



The R-formula rewrites a sum as a single sinusoid

Solve $3\sin\theta + 4\cos\theta = 2$ **for** $0° \leqslant \theta \leqslant 360°$**.** Write LHS as $5\sin(\theta + 53.13°)$: $\sin(\theta + 53.13°) = 0.4$, so $\theta + 53.13° = 23.6°$ or $156.4°$ $(+360°)$. Taking those in range gives $\theta = 103.3°$ or $330.5°$.

2 Practice

2.1 Write $\cos 2\theta$ in terms of $\cos\theta$ only. [1]

2.2 Find R for $7\sin\theta + 24\cos\theta = R\sin(\theta + \alpha)$. [1]

3 Exam-style questions

3.1 Using $\sin 2\theta = 2\sin\theta\cos\theta$, the value of $\sin 2\theta$ when $\sin\theta = \frac{3}{5}$ and $\cos\theta = \frac{4}{5}$ is: [1]

- **A** $\frac{12}{25}$
- **B** $\frac{24}{25}$
- **C** $\frac{7}{25}$
- **D** $\frac{6}{5}$

3.2 (a) Express $4\sin\theta - 3\cos\theta$ in the form $R\sin(\theta - \alpha)$, with $R > 0$ and $0° < \alpha < 90°$. [3]

(b) Hence solve $4\sin\theta - 3\cos\theta = 2$ for $0° \leqslant \theta \leqslant 360°$. [4]

3.3 Prove the identity $\dfrac{\sin 2\theta}{1 + \cos 2\theta} \equiv \tan\theta$. [3]

Solutions

2.1 $\cos 2\theta = 2\cos^2\theta - 1$.

2.2 $R = \sqrt{7^2 + 24^2} = 25$.

3.1 **B**. $2 \times \frac{3}{5} \times \frac{4}{5} = \frac{24}{25}$.

3.2 (a) $R = \sqrt{4^2 + 3^2} = 5$; $\tan\alpha = \frac{3}{4} \Rightarrow \alpha = 36.9°$; $5\sin(\theta - 36.9°)$.
(b) $\sin(\theta - 36.9°) = 0.4 \Rightarrow \theta - 36.9° = 23.6°$ or $156.4°$; $\theta = 60.5°$ or $193.3°$.

3.3 $\sin 2\theta = 2\sin\theta\cos\theta$ and $1 + \cos 2\theta = 2\cos^2\theta$; ratio $= \dfrac{2\sin\theta\cos\theta}{2\cos^2\theta} = \dfrac{\sin\theta}{\cos\theta} = \tan\theta$.

3 **(a)** Express $3\sqrt{2}\sin(x+45°)+\cos x$ in the form $R\cos(x-\alpha)$, where $R>0$ and $0°<\alpha<90°$. [4]

(b) Hence solve the equation $3\sqrt{2}\sin(3\theta + 45°) + \cos 3\theta = -4$ for $0° < \theta < 180°$. [4]

7 **(a)** Express $5\sin\left(x+\frac{1}{6}\pi\right)-4\cos x$ in the form $R\sin(x-\alpha)$, where $R>0$ and $0<\alpha<\frac{1}{2}\pi$. State the exact value of R and give the value of α correct to 3 decimal places. [4]

(b) Hence solve the equation $5\sin\left(2\theta+\frac{1}{6}\pi\right)-4\cos 2\theta=\sqrt{7}$ for $0\leqslant\theta\leqslant\pi$. Give your answers correct to 2 decimal places. [4]

4 Solve the equation $3\cot x - 4\cot 2x = 3$ for $0° \leqslant x \leqslant 180°$. [6]

7 **(a)** Express $7\sin\theta + 24\cos\theta$ in the form $R\cos(\theta - \alpha)$, where $R > 0$ and $0 < \alpha < \frac{1}{2}\pi$. Give the value of α correct to 4 decimal places. [3]

(b) Hence solve the equation $7\sin\frac{1}{3}x + 24\cos\frac{1}{3}x = 24.5$ for $0 < x < \pi$. [4]

3.4 Differentiation (inverse tangent, implicit, parametric)

Name: ______________ Class: __________ Date: ____________ **Total: 14 marks**

KEY WORDS quotient 商 • inverse tangent 反正切 • differentiation 微分

Objective

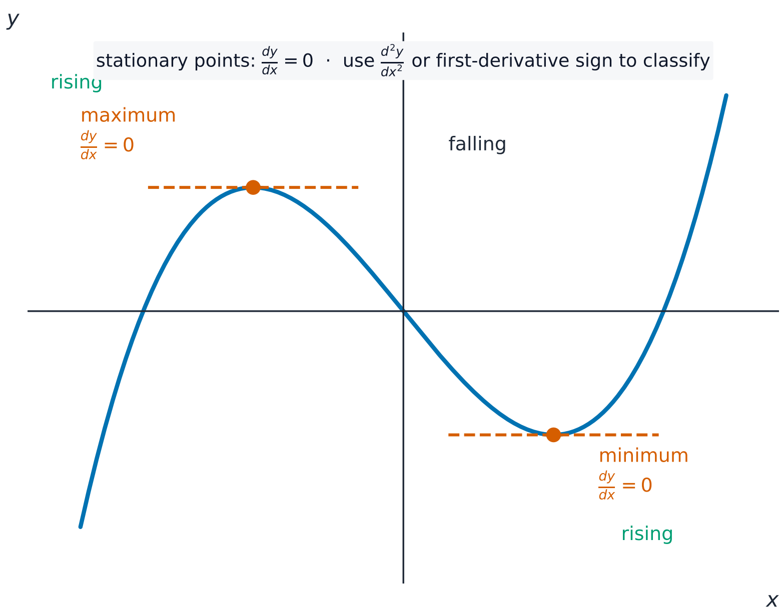
Build the skills to answer exam questions on **differentiation 微分** in Pure 3 — the Pure 2 rules plus the **inverse tangent 反正切** derivative, applied to parametric and implicit curves.

You must be able to:

- use $\dfrac{d}{dx}\tan^{-1}x = \dfrac{1}{1+x^2}$
- apply the product, quotient and chain rules
- differentiate parametric and implicit relations

1 Worked examples

■ Mixed rules



Stationary points and gradients still come from differentiating — now with $\tan^{-1}$ *and implicit methods too*

$y = \tan^{-1}(2x)$**:** chain rule $\to \dfrac{dy}{dx} = \dfrac{1}{1+(2x)^2} \cdot 2 = \dfrac{2}{1+4x^2}$.

Implicit $e^{2y} = x + y$**:** $2e^{2y}\dfrac{dy}{dx} = 1 + \dfrac{dy}{dx} \Rightarrow \dfrac{dy}{dx} = \dfrac{1}{2e^{2y}-1}$.

2 Practice

2.1 Differentiate $y = \tan^{-1}(x^2)$. [2]

2.2 Differentiate $y = x \ln x$. [2]

3 Exam-style questions

3.1 $\dfrac{d}{dx}\tan^{-1}(3x)$ equals: [1]

- **A** $\dfrac{3}{1+9x^2}$
- **B** $\dfrac{1}{1+3x^2}$
- **C** $\dfrac{3}{1+3x^2}$
- **D** $\dfrac{1}{1+9x^2}$

3.2 A curve has parametric equations $x = 1 + 2\cos t$, $y = 3\sin t$ for $0 \leqslant t \leqslant \pi$. Find $\dfrac{dy}{dx}$ in terms of t and the gradient at $t = \frac{\pi}{6}$. [4]

3.3 A curve is given implicitly by $x^2y + y^3 = 10$.

(a) Find $\dfrac{dy}{dx}$ in terms of x and y. [3]

(b) Find the gradient at the point $(1, 2)$. [2]

Solutions

2.1 $\dfrac{dy}{dx} = \dfrac{2x}{1+x^4}$.

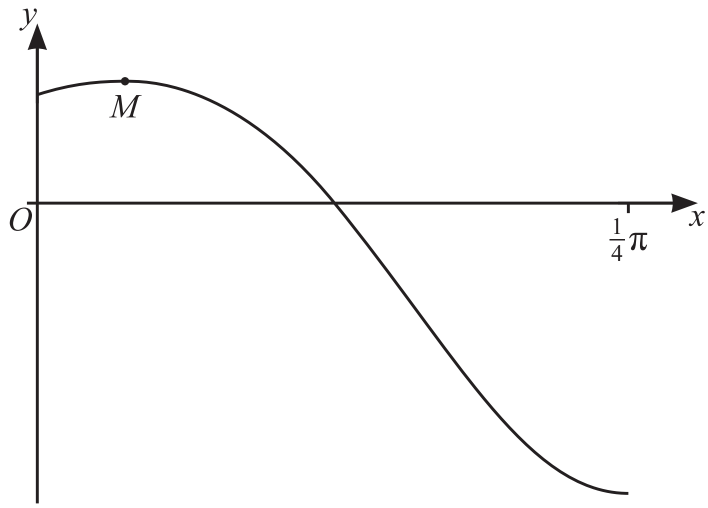
2.2 $\dfrac{dy}{dx} = \ln x + 1$.

3.1 **A**. $\dfrac{1}{1+(3x)^2} \cdot 3 = \dfrac{3}{1+9x^2}$.

3.2 $\dfrac{dx}{dt} = -2\sin t$, $\dfrac{dy}{dt} = 3\cos t$; $\dfrac{dy}{dx} = \dfrac{3\cos t}{-2\sin t} = -\frac{3}{2}\cot t$; at $t = \frac{\pi}{6}$: $-\frac{3}{2}\cot 30^\circ = -\frac{3}{2}\sqrt{3} = -\frac{3\sqrt{3}}{2}$.

3.3 (a) $2xy + x^2\dfrac{dy}{dx} + 3y^2\dfrac{dy}{dx} = 0$; $\dfrac{dy}{dx} = -\dfrac{2xy}{x^2+3y^2}$.
(b) at $(1, 2)$: $-\dfrac{2(1)(2)}{1+12} = -\dfrac{4}{13}$.

4



The diagram shows the graph of $y = e^{\sin 2x}\cos 4x$ for $0 \leqslant x \leqslant \frac{1}{4}\pi$, and its maximum point M.

Find the x-coordinate of M. [5]

7 The parametric equations of a curve are

$$x = t^2 - \ln(2t+1), \quad y = \frac{t}{2t+1}.$$

Obtain a simplified expression for $\frac{dy}{dx}$ in terms of t. [5]

4 The parametric equations of a curve are

$$x = e^{\tan t}, \quad y = 3\tan^2 t.$$

Find the equation of the tangent to the curve at the point (e, 3). Give your answer in the form $y = mx + c$, where m and c are exact. [6]

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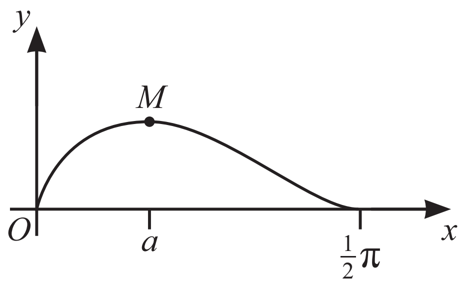

The diagram shows the curve $y = \cos x\sqrt{\sin 2x}$ for $0 \leqslant x \leqslant \frac{1}{2}\pi$. The curve has a maximum point at M, where $x = a$.

(a) Find the exact value of a. [6]

(b) The region enclosed between the x-axis and the curve is rotated through 2π radians about the x-axis.

Find the exact volume of the solid generated. [5]

Additional page

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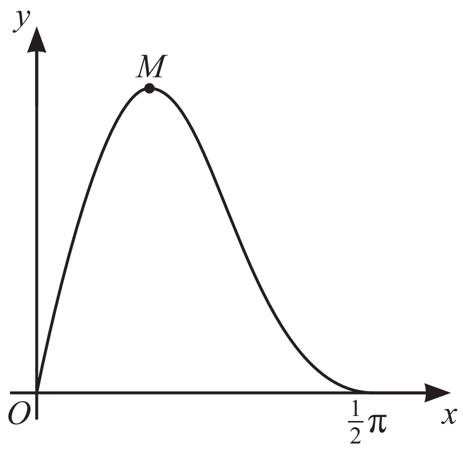


The diagram shows the graph of $y = 5\sin 2x\cos^2 x$ for $0 \leqslant x \leqslant \frac{1}{2}\pi$ and its maximum point M.

(a) Find the exact x-coordinate of M. [6]

(b) By using the substitution $u = \cos x$, find the area of the region bounded by the curve, the x-axis between $x = 0$ and $x = \frac{1}{4}\pi$, and the line $x = \frac{1}{4}\pi$. [5]

Additional page

If you use the following page to complete the answer to any question, the question number must be clearly shown.

3.5 Integration (by parts, substitution, partial fractions)

Name: ____________ Class: __________ Date: ____________ **Total: 16 marks**

KEY WORDS substitution 换元积分 • integration 积分 • integration by parts 分部积分 • partial fractions 部分分式 • rational function 有理函数

Objective

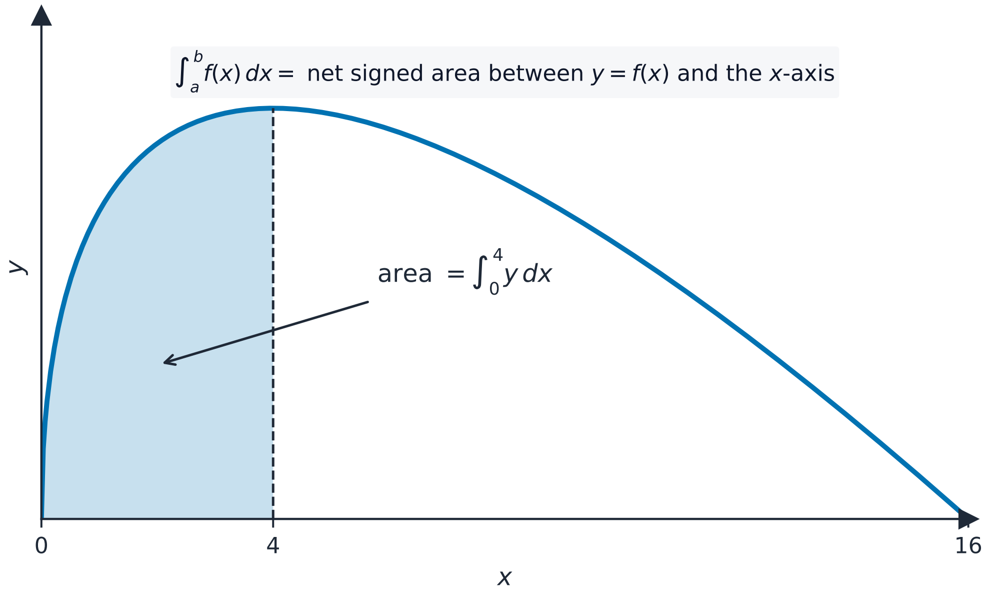
Build the skills to answer exam questions on **integration 积分** in Pure 3 — the standard integral $\dfrac{1}{x^2+a^2}$, **integration by parts 分部积分**, **substitution 换元积分**, partial fractions, and the $\dfrac{kf'(x)}{f(x)}$ pattern.

You must be able to:

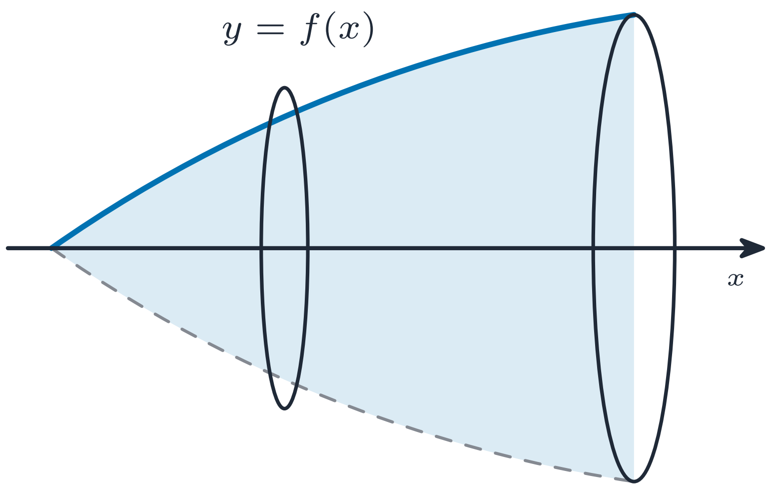
- recognise $\int \frac{1}{x^2+a^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$ and $\int \frac{kf'(x)}{f(x)}\,dx = k\ln|f(x)| + C$
- integrate by parts and by a given substitution
- integrate a rational function via partial fractions

1 Worked examples

■ By parts & the log pattern



A definite integral is still an area; Pure 3 adds the methods to evaluate harder ones



An application of integration: a volume of revolution

$\int x \cos x \, dx$ — by parts ($u = x$, $dv = \cos x \, dx$): $= x \sin x - \int \sin x \, dx = x \sin x + \cos x + C$.

$\int \frac{2x}{x^2 + 1} \, dx = \ln(x^2 + 1) + C$ (top is the derivative of the bottom).

2 Practice

2.1 Find $\int \frac{1}{x^2+9}\, dx$. [2]

2.2 Find $\int \frac{3x^2}{x^3+1}\, dx$. [2]

3 Exam-style questions

3.1 $\int xe^x\, dx$ equals: [1]

- **A** $xe^x - e^x + C$
- **B** $xe^x + e^x + C$
- **C** $\frac{1}{2}x^2e^x + C$
- **D** $e^x + C$

3.2 Use the substitution $u = 1 + x^2$ to find $\int_0^2 x\sqrt{1+x^2}\, dx$. [5]

3.3 (a) Express $\frac{4}{(x)(x+2)}$ in partial fractions. [3]

(b) Hence find $\int_1^3 \frac{4}{x(x+2)}\,dx$, giving an exact answer. [3]

Solutions

2.1 $\frac{1}{3}\tan^{-1}\dfrac{x}{3} + C$.

2.2 $\ln|x^3+1| + C$.

3.1 **A**. By parts: $xe^x - \int e^x\,dx = xe^x - e^x + C$.

3.2 $u = 1 + x^2 \Rightarrow du = 2x\,dx$, so $x\,dx = \frac{1}{2}du$; limits $x = 0 \to u = 1$, $x = 2 \to u = 5$; $\displaystyle\int_1^5 \tfrac{1}{2}\sqrt{u}\,du = \tfrac{1}{2}\left[\tfrac{2}{3}u^{3/2}\right]_1^5 = \tfrac{1}{3}(5^{3/2}-1)$; $= \frac{1}{3}(11.180-1) = 3.39$.

3.3 (a) $4 = A(x+2) + Bx$; $x = 0 \Rightarrow A = 2$; $x = -2 \Rightarrow B = -2$; $\dfrac{2}{x} - \dfrac{2}{x+2}$.

(b) $\displaystyle\int_1^3 \left(\frac{2}{x} - \frac{2}{x+2}\right)dx = \Big[2\ln x - 2\ln(x+2)\Big]_1^3 = 2\ln\frac{x}{x+2}\Big|_1^3$; $= 2\left(\ln\frac{3}{5} - \ln\frac{1}{3}\right) = 2\ln\frac{9}{5}$.

1 Find the exact value of $\int_1 \ln 3x \, dx$. Give your answer in the form $a + \ln b$, where a and b are integers. [4]

9 The constant a is such that $\int_{1} 6x \ln x \, \mathrm{d}x = 4$.

(a) Show that $a = \exp\left(\frac{1}{6}\left(\frac{5}{a^2} + 3\right)\right)$, where $\exp(x)$ denotes e^x. [5]

(b) Verify by calculation that a lies between 2 and 2.1. [2]

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(c) Use an iterative formula based on the equation in part **(a)** to determine a correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

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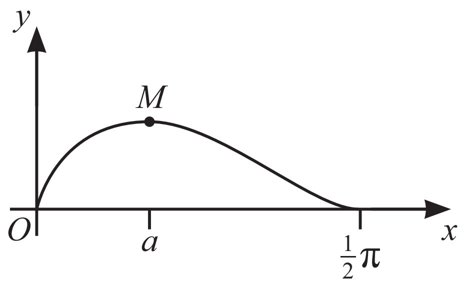

The diagram shows the curve $y = \cos x\sqrt{\sin 2x}$ for $0 \leqslant x \leqslant \frac{1}{2}\pi$. The curve has a maximum point at M, where $x = a$.

(a) Find the exact value of a. [6]

(b) The region enclosed between the x-axis and the curve is rotated through 2π radians about the x-axis.

Find the exact volume of the solid generated. [5]

Additional page

If you use the following page to complete the answer to any question, the question number must be clearly shown.

10 **(a)** Find the quotient and remainder when x^2 is divided by $1+4x^2$. [2]

(b) Find the exact value of $\int_0^{0.5} x\tan^{-1}(2x)\,\mathrm{d}x$. [6]

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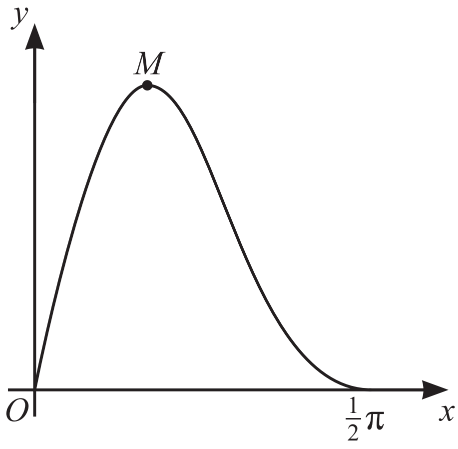


The diagram shows the graph of $y = 5\sin 2x\cos^2 x$ for $0 \leqslant x \leqslant \frac{1}{2}\pi$ and its maximum point M.

(a) Find the exact x-coordinate of M. [6]

(b) By using the substitution $u = \cos x$, find the area of the region bounded by the curve, the x-axis between $x = 0$ and $x = \frac{1}{4}\pi$, and the line $x = \frac{1}{4}\pi$. [5]

Additional page

If you use the following page to complete the answer to any question, the question number must be clearly shown.

3.6 Numerical solution of equations

Name: ____________ Class: ________ Date: __________ **Total: 13 marks**

KEY WORDS iteration 迭代 • converges 收敛 • sign change 变号 • numerical solution of equations 方程的数值解

Objective

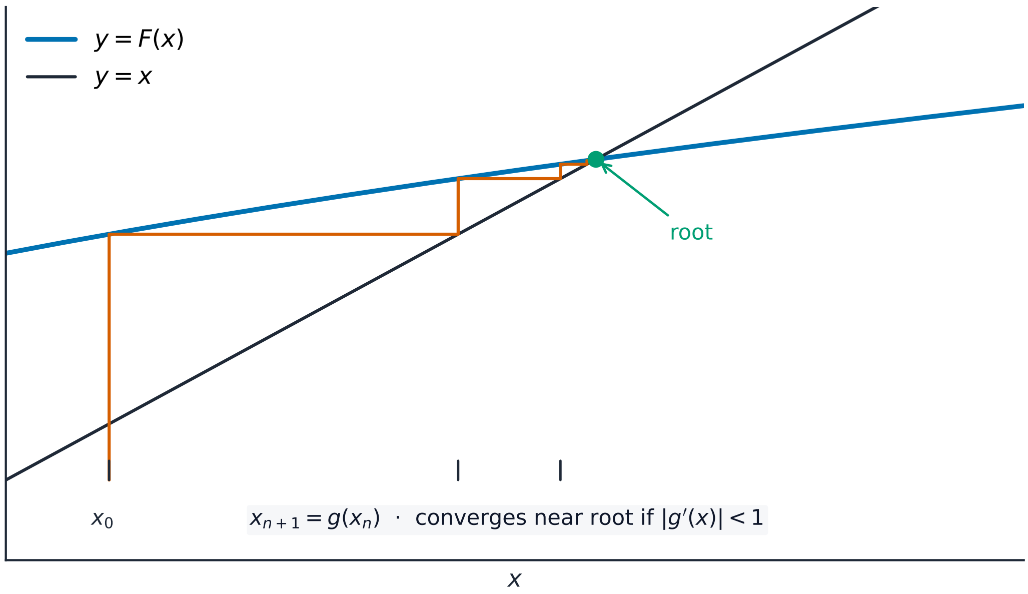
Build the skills to answer exam questions on the **numerical solution of equations** 方程的数值解 in Pure 3 — locating a root by a **sign change** 变号 and finding it by **iteration** 迭代, often after some algebra/calculus.

You must be able to:

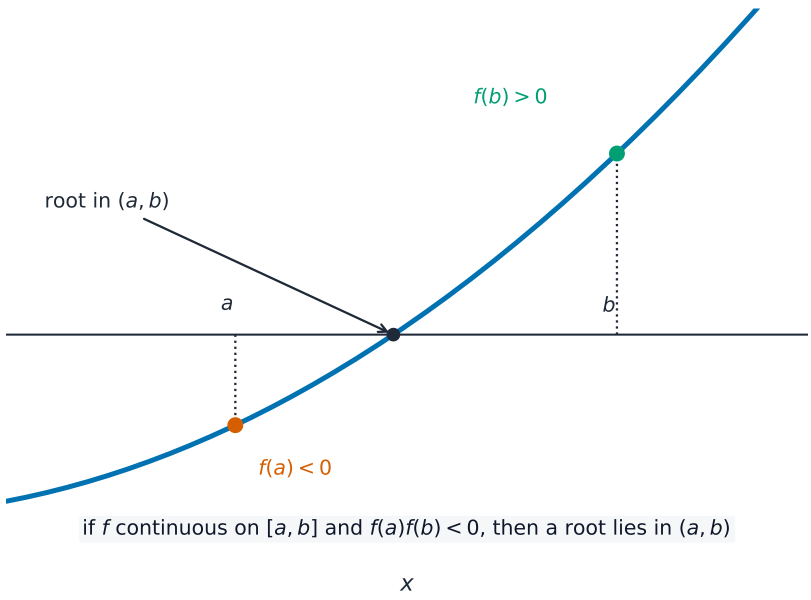
- show a root lies in an interval by a sign change
- show an equation rearranges into a given iterative form
- iterate $x_{n+1} = F(x_n)$ to the required accuracy

1 Worked examples

Iteration to a root



Each step: up to $y = F(x)$, across to $y = x$ — the iteration converges to the root



Locate the root first by a sign change

$x_{n+1} = \sqrt[3]{-2x_n - 4.5}$ **from** $x_0 = -1.2$**:** $x_1 = -1.281$, $x_2 = -1.247$, ... $\to$ root ≈ -1.26 (3 s.f.).

2 Practice

2.1 $f(x) = e^x - 3x$. Show that a root lies between $x = 1$ and $x = 2$. [2]

2.2 Rearrange $x^3 + x - 5 = 0$ into the form $x = F(x)$ suitable for iteration. [1]

3 Exam-style questions

3.1 An iteration $x_{n+1} = F(x_n)$ from a suitable start **converges** to a root if the values: [1]

- **A** grow without limit
- **B** settle towards a fixed number
- **C** alternate between two values forever
- **D** stay equal to x_0

3.2 The equation $x^3 = 4x + 1$ has a root α near $x = 2$.

(a) Verify there is a root between $x = 2$ and $x = 2.2$. [2]

(b) Show the equation can be written $x = \sqrt[3]{4x+1}$ and use $x_{n+1} = \sqrt[3]{4x_n + 1}$, $x_0 = 2.1$, to find α to 2 d.p. [4]

3.3 By sketching $y = e^{-x}$ and $y = x$ on the same axes, state the number of roots of $e^{-x} = x$, and give an interval of width 0.5 containing the root. [3]

Solutions

2.1 $f(1) = e - 3 = -0.282$, $f(2) = e^2 - 6 = 1.389$; opposite signs $\to$ a root lies between 1 and 2.

2.2 e.g. $x = \sqrt[3]{5 - x}$.

3.1 B. Convergence means the iterates settle towards a fixed value.

3.2 (a) $f(x) = x^3 - 4x - 1$: $f(2) = -1 < 0$, $f(2.2) = 10.648 - 8.8 - 1 = 0.848 > 0$; sign change $\to$ root between.
(b) $x^3 = 4x + 1 \Rightarrow x = \sqrt[3]{4x + 1}$; $x_1 = \sqrt[3]{9.4} = 2.110$, $x_2 = 2.113$, $x_3 = 2.114$; $\alpha = 2.11$.

3.3 the decreasing curve $y = e^{-x}$ meets the line $y = x$ exactly **once**, so there is 1 root; $g(x) = e^{-x} - x$ has $g(0.5) = 0.607 - 0.5 > 0$ and $g(1) = 0.368 - 1 < 0$, so the root lies between 0.5 and 1.

9 **(a)** By sketching a suitable pair of graphs, show that the equation

$$\sec 2x = -\mathrm{e}^{x}$$

has only one root in the interval $0 < x < \frac{1}{2}\pi$. [2]

(b) Show by calculation that this root lies between 0.9 and 1. [2]

(c) Show that if a sequence of values given by the iterative formula

$$x_{n+1} = \tfrac{1}{2}\cos^{-1}\left(-e^{-x_n}\right)$$

converges, then it converges to the root of the equation in part **(a)**. [1]

(d) Use the iterative formula given in part **(c)** to calculate x correct to 3 decimal places. Give the result of each iteration to 5 decimal places. [3]

9 The constant a is such that $\int_1 6x \ln x \, \mathrm{d}x = 4$.

(a) Show that $a = \exp\left(\frac{1}{6}\left(\frac{5}{a^2} + 3\right)\right)$, where $\exp(x)$ denotes e^x. [5]

(b) Verify by calculation that a lies between 2 and 2.1. [2]

(c) Use an iterative formula based on the equation in part **(a)** to determine a correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

6 **(a)** By sketching a suitable pair of graphs, show that the equation

$$|x-2| = 2\sin\tfrac{1}{2}x$$

has only one root in the interval $0 < x < \pi$. [2]

(b) Show by calculation that this root lies between 1 and 1.5. [2]

(c) Use the iterative formula $x_{n+1} = 2 - 2\sin\frac{1}{2}x_n$ with an initial value of 1.03 to calculate the root correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

3.7 Vectors

Name: ______________ Class: _________ Date: ___________ **Total: 16 marks**

KEY WORDS vectors 向量 • position vector 位置向量 • scalar product 数量积 • scalar 标量 • magnitude 模长

Objective

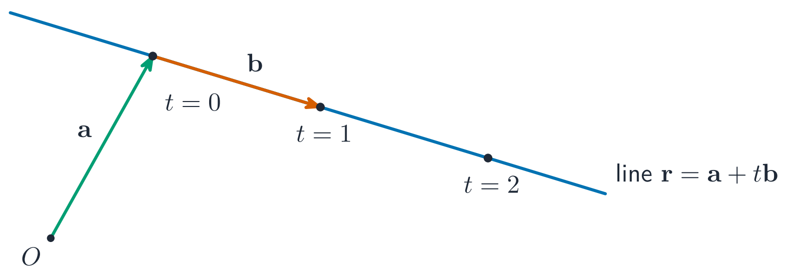
Build the skills to answer exam questions on **vectors 向量** — magnitude and unit vectors, the **vector equation of a line 直线向量方程**, the **scalar product 数量积**, and the angle between lines.

You must be able to:

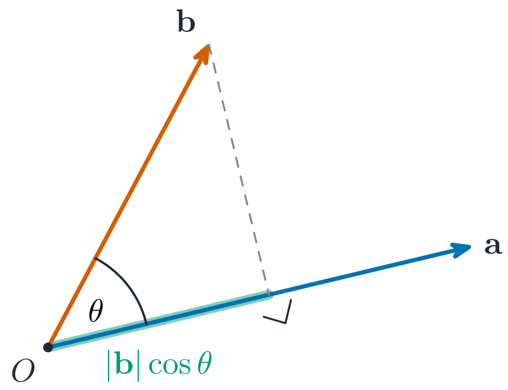
- find magnitude, a unit vector and a displacement $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$
- write and use $\mathbf{r} = \mathbf{a} + t\mathbf{b}$; test for intersection
- use $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta$ to find angles / perpendicularity

1 Worked examples

Line and scalar product



$\mathbf{r} = \mathbf{a} + t\mathbf{b}$: *start at* $\mathbf{a}$, *add* t *copies of the direction* $\mathbf{b}$



$\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$*; it is* 0 *when the vectors are perpendicular*

Angle between $\mathbf{a} = \mathbf{i}+2\mathbf{j}+2\mathbf{k}$, $\mathbf{b} = 2\mathbf{i}+2\mathbf{j}+\mathbf{k}$: $\mathbf{a}\cdot\mathbf{b} = 8$, $|\mathbf{a}| = |\mathbf{b}| = 3$, so $\cos\theta = \frac{8}{9}$, $\theta = 27.3°$.

2 Practice

2.1 Find $|\mathbf{v}|$ for $\mathbf{v} = 3\mathbf{i} - 4\mathbf{j} + 12\mathbf{k}$. [2]

2.2 Find $\mathbf{a}\cdot\mathbf{b}$ for $\mathbf{a} = 2\mathbf{i} - \mathbf{j} + \mathbf{k}$, $\mathbf{b} = \mathbf{i} + 3\mathbf{j} + \mathbf{k}$. [2]

3 Exam-style questions

3.1 Two vectors are perpendicular when their scalar product is: [1]

- **A** 1
- **B** 0
- **C** the product of their magnitudes
- **D** undefined

3.2 The line l has equation $\mathbf{r} = \begin{pmatrix} 1 \\ 0 \\ 2 \end{pmatrix} + t \begin{pmatrix} 2 \\ 1 \\ -1 \end{pmatrix}$.

(a) Find the position vector of the point on l where $t = 3$. [2]

(b) Find the acute angle between l and the direction $\mathbf{d} = \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix}$. [4]

3.3 Points $A(1, 2, 1)$, $B(3, 3, 4)$ and $C(2, 0, 3)$ are given.

(a) Find $\overrightarrow{AB}$ and $\overrightarrow{AC}$. [2]

(b) Find angle BAC. [3]

Solutions

2.1 $|\mathbf{v}| = \sqrt{9+16+144} = \sqrt{169} = 13$.

2.2 $\mathbf{a}\cdot\mathbf{b} = 2-3+1 = 0$.

3.1 **B**. Perpendicular vectors have scalar product 0.

3.2 (a) $\begin{pmatrix}1\\0\\2\end{pmatrix} + 3\begin{pmatrix}2\\1\\-1\end{pmatrix} = \begin{pmatrix}7\\3\\-1\end{pmatrix}$.

(b) direction of l is $(2,1,-1)$; $\mathbf{b}\cdot\mathbf{d} = 2+2-2 = 2$; $|\mathbf{b}| = \sqrt{6}$, $|\mathbf{d}| = 3$; $\cos\theta = \dfrac{2}{3\sqrt{6}} = 0.2722 \Rightarrow \theta = 74.2°$.

3.3 (a) $\overrightarrow{AB} = (2,1,3)$, $\overrightarrow{AC} = (1,-2,2)$.

(b) $\overrightarrow{AB}\cdot\overrightarrow{AC} = 2-2+6 = 6$; $|\overrightarrow{AB}| = \sqrt{14}$, $|\overrightarrow{AC}| = 3$; $\cos(BAC) = \dfrac{6}{3\sqrt{14}} = 0.5345 \Rightarrow BAC = 57.7°$.

11 With respect to the origin O, the points A, B, C and D have position vectors given by

$$\overrightarrow{OA} = \begin{pmatrix} 1 \\ 5 \\ 3 \end{pmatrix}, \qquad \overrightarrow{OB} = \begin{pmatrix} 0 \\ 4 \\ 1 \end{pmatrix}, \qquad \overrightarrow{OC} = \begin{pmatrix} 1 \\ -3 \\ 1 \end{pmatrix} \qquad \text{and} \qquad \overrightarrow{OD} = \begin{pmatrix} 3 \\ -5 \\ 4 \end{pmatrix}.$$

The line m passes through the points A and B.

(a) Find a vector equation for m. [2]

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(b) Find the position vector of the point of intersection of m and the line passing through the points C and D. [4]

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(c) Find the position vector of the foot of the perpendicular from C to m. [4]

Additional page

If you use the following page to complete the answer to any question, the question number must be clearly shown.

8 With respect to the origin O, the points A and B have position vectors $2\mathbf{i}+4\mathbf{k}$ and $5\mathbf{i}+\mathbf{j}+6\mathbf{k}$ respectively. The line l_1 passes through the points A and B.

(a) Find a vector equation for the line l_1. [2]

The line l_2 has equation $\mathbf{r} = 2\mathbf{i}+\mathbf{j}+5\mathbf{k}+\mu(\mathbf{i}+2\mathbf{j}+3\mathbf{k})$.

(b) Show that l_1 and l_2 do **not** intersect. [4]

(c) Find the acute angle between the directions of l_1 and l_2. [3]

9 With respect to the origin O, the points A, B and C have position vectors given by

$$\overrightarrow{OA} = \begin{pmatrix} 1 \\ -4 \\ 2 \end{pmatrix}, \qquad \overrightarrow{OB} = \begin{pmatrix} -2 \\ 1 \\ 3 \end{pmatrix} \qquad \text{and} \qquad \overrightarrow{OC} = \begin{pmatrix} 2 \\ 3 \\ 5 \end{pmatrix}.$$

(a) Find a vector equation for the line through A and B. [2]

(b) Using a scalar product, find the exact value of $\cos BAC$. [4]

(c) Hence find the exact area of triangle ABC. [3]

A-LEVEL MATHEMATICS • EXERCISE SHEET

3.8 Differential equations

Name: ____________ Class: ________ Date: __________ **Total: 15 marks**

KEY WORDS general solution 通解 • initial condition 初始条件 • particular solution 特解 • differential equations 微
• separable 可分离变量

Objective

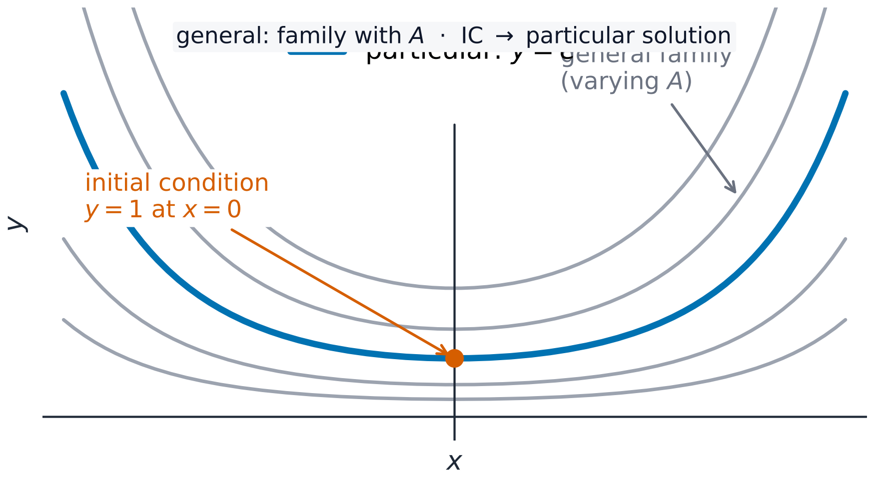
Build the skills to answer exam questions on **differential equations 微分方程** — solving a **separable 可分离变量** first-order equation, finding the **general 通解** and **particular solution 特解**, and setting up a model.

You must be able to:

- separate variables and integrate both sides
- use an initial condition to find the constant
- form and solve a differential equation from a rate description

1 Worked examples

■ Separate and solve



The constant gives a family of curves; an initial condition picks one particular solution

Solve $\dfrac{dy}{dx} = xy$, $y = 1$ **at** $x = 0$.

$$\int \frac{1}{y}\,dy = \int x\,dx \Rightarrow \ln y = \tfrac{1}{2}x^2 + c \Rightarrow y = Ae^{x^2/2}; \quad y(0) = 1 \Rightarrow A = 1, \text{ so } y = e^{x^2/2}.$$

2 Practice

2.1 Separate the variables for $\dfrac{dy}{dx} = \dfrac{y}{x}$ (write each side ready to integrate). [1]

2.2 Solve $\dfrac{dy}{dx} = 3y$, giving the general solution. [2]

3 Exam-style questions

3.1 The general solution of $\dfrac{dy}{dx} = ky$ is: [1]

- **A** $y = kx + c$
- **B** $y = Ae^{kx}$
- **C** $y = Ax^k$
- **D** $y = \frac{1}{2}kx^2 + c$

3.2 Solve $\dfrac{dy}{dx} = 2x(y+1)$, given $y = 0$ when $x = 0$. Give y explicitly. [5]

3.3 A tank loses water so that the depth h falls at a rate proportional to $\sqrt{h}$: $\dfrac{dh}{dt} = -k\sqrt{h}$. Initially $h = H$.

(a) Solve to show $\sqrt{h} = \sqrt{H} - \frac{1}{2}kt$. [4]

(b) Find, in terms of H and k, the time for the tank to empty. [2]

Solutions

2.1 $\dfrac{1}{y}\,dy = \dfrac{1}{x}\,dx$.

2.2 $\ln y = 3x + c \Rightarrow y = Ae^{3x}$.

3.1 B. $\dfrac{dy}{dx} = ky$ gives $y = Ae^{kx}$.

3.2 $\displaystyle\int \frac{1}{y+1}\,dy = \int 2x\,dx$; $\ln(y+1) = x^2 + c$; $y(0) = 0 \Rightarrow c = 0$; $y+1 = e^{x^2}$; $y = e^{x^2} - 1$.

3.3 (a) $\displaystyle\int h^{-1/2}\,dh = \int -k\,dt$; $2\sqrt{h} = -kt + c$; $h = H$ at $t = 0 \Rightarrow c = 2\sqrt{H}$; $\sqrt{h} = \sqrt{H} - \tfrac{1}{2}kt$.

(b) empty when $h = 0$: $0 = \sqrt{H} - \tfrac{1}{2}kt \Rightarrow t = \dfrac{2\sqrt{H}}{k}$.

8 The variables x and y satisfy the differential equation

$$\left(x^2+1\right)\frac{\mathrm{d}y}{\mathrm{d}x}=kx\mathrm{e}^{2y},$$

where k is a constant. It is given that $y=0$ when $x=0$ and that $y=-\frac{1}{2}$ when $x=1$.

Solve the differential equation and find the exact value of y when $x=\sqrt{3}$. [8]

10 **(a)** Find the quotient and remainder when $x^3 + 5x^2 - 2x - 15$ is divided by $x^2 - 3$. [3]

(b) The variables x and y satisfy the differential equation

$$\frac{dy}{dx} = \frac{x^3 + 5x^2 - 2x - 15}{6y(x^2 - 3)}.$$

It is given that $y = 2$ when $x = 2$.

Solve the differential equation to obtain an expression for y^2 in terms of x. [5]

8 The variables x and θ satisfy the differential equation

$$\sin 2\theta \frac{dx}{d\theta} = (4x+3)\cos 2\theta,$$

and $x = 0$ when $\theta = \frac{1}{12}\pi$.

Solve the differential equation and obtain an expression for x in terms of θ. [7]

A-LEVEL MATHEMATICS • EXERCISE SHEET

3.9 Complex numbers

Name: ____________ Class: ________ Date: __________ **Total: 14 marks**

KEY WORDS complex numbers 复数 • argand diagram 阿干图 • argument 辐角 • loci 轨迹 • modulus 模

Objective

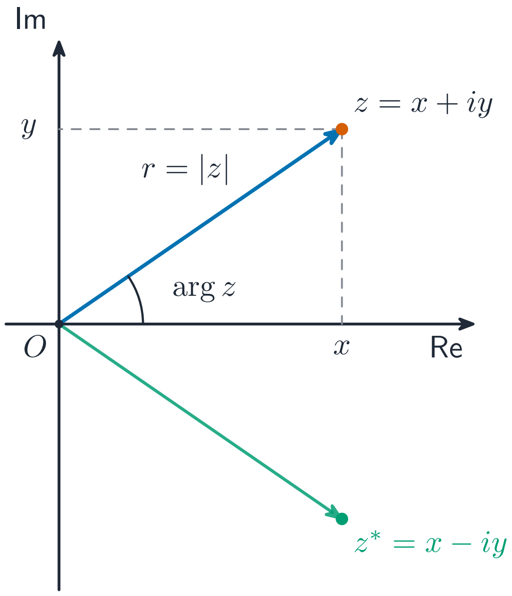
Build the skills to answer exam questions on **complex numbers 复数** — arithmetic and **conjugates 共轭**, **modulus 模** and **argument 辐角**, the **Argand diagram 阿干图**, and **loci 轨迹**.

You must be able to:

- add, multiply and divide complex numbers (divide using the conjugate)
- find modulus and argument; convert to polar form
- sketch loci such as $|z-a|=r$, $|z-a|=|z-b|$, $\arg(z-a)=\theta$

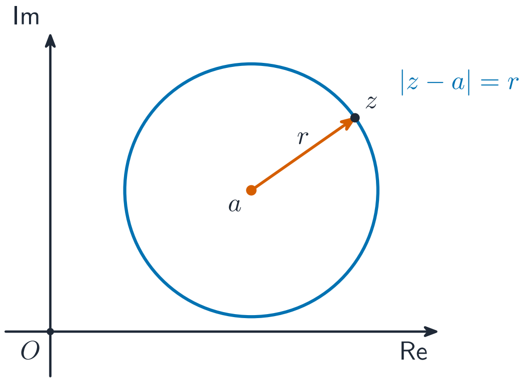
1 Worked examples

■ Division & loci



$|z|$ is the distance from O, $\arg z$ the angle, z^* the reflection in the real axis

$$\frac{3+i}{1-i} = \frac{(3+i)(1+i)}{2} = \frac{2+4i}{2} = 1+2i.$$



$|z-a| = r$ is a circle of radius r centred at a

2 Practice

2.1 Find the modulus of $z = 3 + 4i$. [1]

2.2 Write $(2+i)(3-i)$ in the form $x + iy$. [2]

3 Exam-style questions

3.1 The argument of $z = -1 + i$ (in radians, $-\pi < \arg z \leqslant \pi$) is: [1]

- **A** $\frac{\pi}{4}$
- **B** $\frac{3\pi}{4}$
- **C** $-\frac{\pi}{4}$
- **D** $\frac{\pi}{2}$

3.2 The complex number z satisfies $z = \dfrac{10}{2-i}$.

(a) Express z in the form $x + iy$. [3]

(b) Find $|z|$ and $\arg z$ (in radians). [3]

3.3 On a single Argand diagram, sketch the locus of points z for which:

(a) $|z-(2+i)|=2$. [2]

(b) Describe the locus $|z-3|=|z+1|$. [2]

Solutions

2.1 $|z| = \sqrt{3^2 + 4^2} = 5$.

2.2 $(2+i)(3-i) = 6 - 2i + 3i - i^2 = 7 + i$.

3.1 B. The point $(-1, 1)$ is in the second quadrant: $\arg z = \pi - \frac{\pi}{4} = \frac{3\pi}{4}$.

3.2 (a) $z = \dfrac{10(2+i)}{(2-i)(2+i)} = \dfrac{20+10i}{5} = 4 + 2i$.
(b) $|z| = \sqrt{4^2 + 2^2} = \sqrt{20} = 2\sqrt{5}$; $\arg z = \tan^{-1} \frac{2}{4} = 0.4636$ rad.

3.3 (a) a circle of radius 2 centred at the point $(2, 1)$.
(b) the perpendicular bisector of the segment joining 3 and -1 — i.e. the vertical line $x = 1$.

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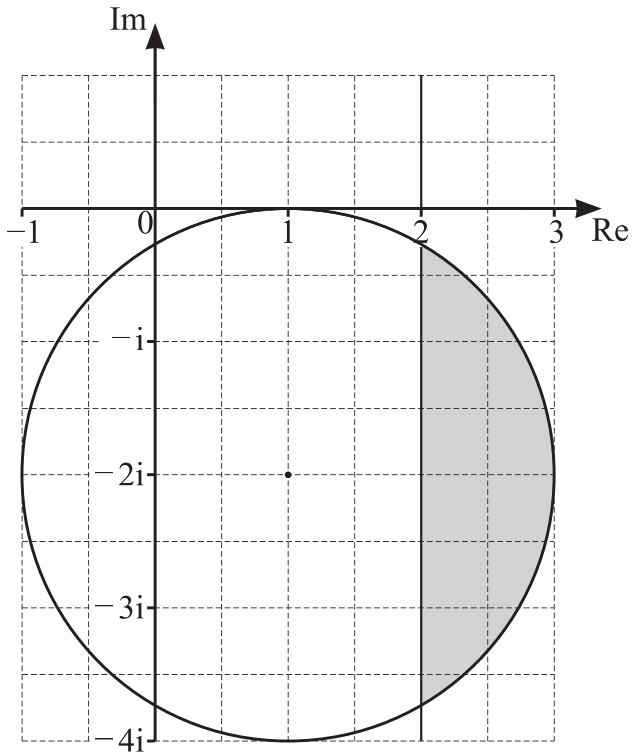

The shaded region on the Argand diagram shows points representing complex numbers z defined by two inequalities. The shaded region is bounded by a circle and a line parallel to the imaginary axis. The boundaries of the region are included in the shaded region.

(a) Find two inequalities in terms of z that define the shaded region. [3]

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(b) Calculate the least value of $\arg z$ for points in this region. [3]

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6 Solve the quadratic equation $(2+\mathrm{i})w^2+4w+2-\mathrm{i}=0$. Give your answers in the form $x+\mathrm{i}y$, where x and y are real. [5]

3 Find the complex numbers z for which $\dfrac{z+5i}{z-5}$ is real and $|z| = \sqrt{17}$. Give your answers in the form $z = x + iy$, where x and y are real. [6]

6 It is given that $z_1 = 3\mathrm{e}^{\frac{1}{4}\pi\mathrm{i}}$, $z_2 = \frac{3}{2}\mathrm{e}^{\frac{1}{6}\pi\mathrm{i}}$ and $\omega = 2\mathrm{e}^{\frac{1}{2}\pi\mathrm{i}}$.

(a) State the values of ωz_1 and ωz_2. Give your answers in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r > 0$ and $-\pi < \theta \leqslant \pi$. [2]

(b) On a sketch of an Argand diagram with origin O, show the points A, B, C and D representing the complex numbers z_1, z_2, ωz_1 and ωz_2 respectively. [2]

(c) State the geometric effects of multiplying z_1 and z_2 by ω. [2]

3 On an Argand diagram shade the region whose points represent complex numbers z which satisfy both the inequalities $|z-3\mathrm{i}| \leqslant 2$ and $\frac{1}{4}\pi \leqslant \arg(z-1-2\mathrm{i}) \leqslant \frac{3}{4}\pi$. [5]

5 The square roots of $-1-4\sqrt{5}\,\mathrm{i}$ can be expressed in the Cartesian form $x+\mathrm{i}y$, where x and y are real and exact.

By first forming a quartic equation in x or y, find the square roots of $-1-4\sqrt{5}\,\mathrm{i}$ in exact Cartesian form. [5]