

6.3 Continuous random variables

Name: _____ Class: _____ Date: _____

Total: 17 marks

KEY WORDS probability density function 概率密度函数 • median 中位数 • continuous random variable 连续型随机变量
 • cumulative distribution function 累积分布函数 • percentiles 百分位数

Objective

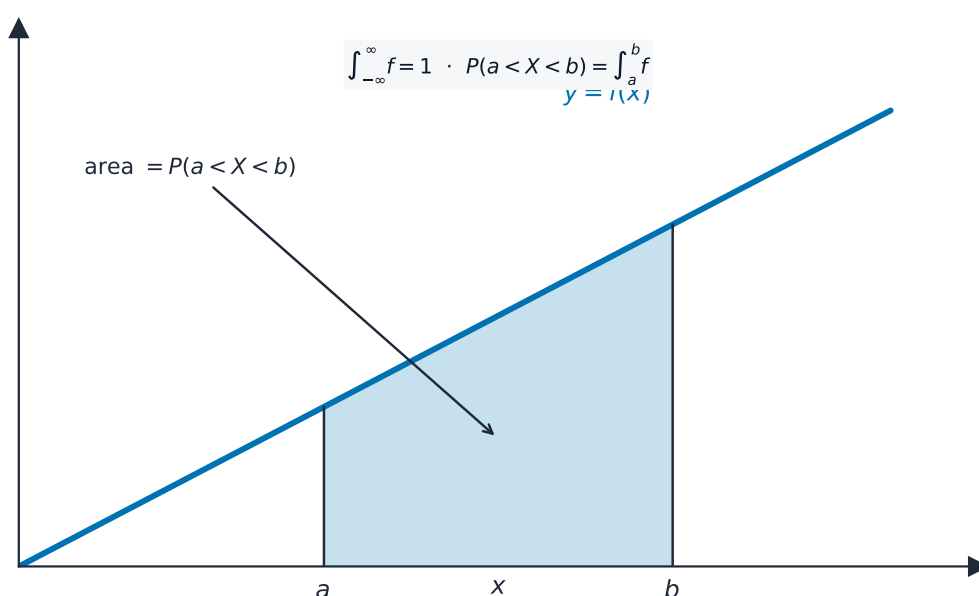
Build the skills to answer exam questions on **continuous random variables** 连续型随机变量 — the **probability density function** 概率密度函数 $f(x)$, finding probabilities and $E(X)$, and the **cumulative distribution function** 累积分布函数 $F(x)$.

You must be able to:

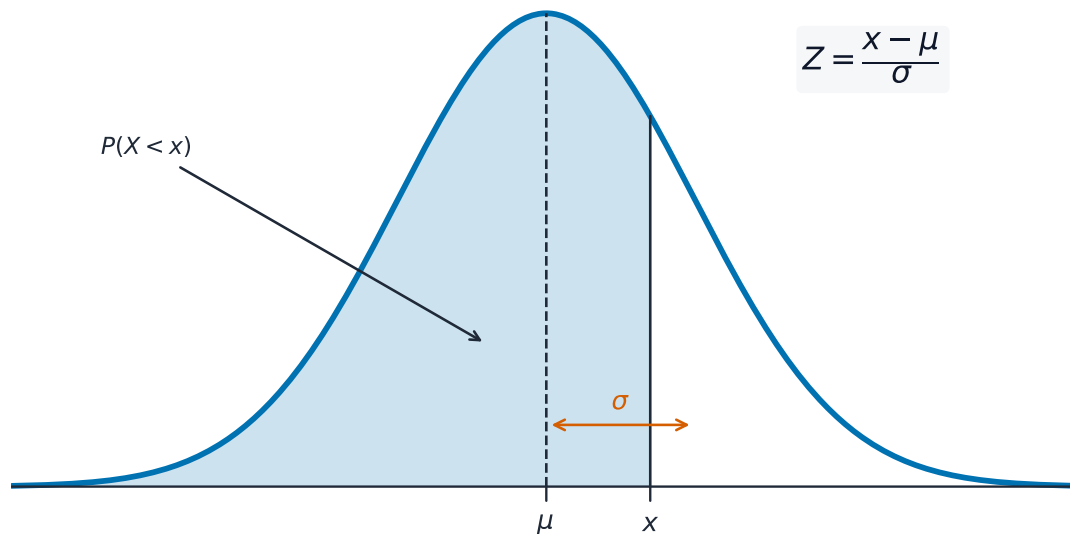
- use $\int f(x) dx = 1$ to find a constant; find $P(a < X < b) = \int_a^b f dx$
- find $E(X) = \int xf(x) dx$ (and $\text{Var}(X)$)
- find $F(x)$ and use it for the median/percentiles

1 Worked examples

■ Density and expectation



$P(a < X < b)$ is the area under $f(x)$ between a and b ; the total area is 1



The normal distribution is a continuous random variable

$$f(x) = \frac{1}{2}x \text{ for } 0 \leq x \leq 2: E(X) = \int_0^2 x \cdot \frac{1}{2}x \, dx = \left[\frac{x^3}{6} \right]_0^2 = \frac{8}{6} = \frac{4}{3}.$$

Median m : solve $F(m) = 0.5$.

2 Practice

2.1 $f(x) = kx$ for $0 \leq x \leq 2$. Show that $k = \frac{1}{2}$. [2]

2.2 For $f(x) = \frac{1}{2}x$ on $[0, 2]$, find $P(X < 1)$. [2]

3 Exam-style questions

3.1 For a valid pdf, $\int_{-\infty}^{\infty} f(x) \, dx$ must equal: [1]

- A 0
- B 0.5
- C 1
- D ∞

3.2 A continuous random variable has $f(x) = \frac{3}{4}(1-x^2)$ for $-1 \leq x \leq 1$ (and 0 elsewhere).

(a) Verify that $\int_{-1}^1 f(x) dx = 1$. [3]

(b) Find $P(X > 0.5)$. [3]

3.3 A variable has $f(x) = \frac{3}{8}x^2$ for $0 \leq x \leq 2$.

(a) Find $E(X)$. [3]

(b) Find the median m . [3]

Solutions

$$\mathbf{2.1} \quad \int_0^2 kx \, dx = \left[\frac{kx^2}{2} \right]_0^2 = 2k = 1 \Rightarrow k = \frac{1}{2}.$$

$$\mathbf{2.2} \quad \int_0^1 \frac{1}{2}x \, dx = \left[\frac{x^2}{4} \right]_0^1 = \frac{1}{4}.$$

3.1 C. The total area under a pdf is 1.

$$\mathbf{3.2} \quad (\text{a}) \quad \int_{-1}^1 \frac{3}{4}(1 - x^2) \, dx = \frac{3}{4} \left[x - \frac{x^3}{3} \right]_{-1}^1 = \frac{3}{4} \left(\frac{2}{3} - \left(-\frac{2}{3}\right) \right) = \frac{3}{4} \cdot \frac{4}{3} = 1.$$

$$(\text{b}) \quad \int_{0.5}^1 \frac{3}{4}(1 - x^2) \, dx = \frac{3}{4} \left[x - \frac{x^3}{3} \right]_{0.5}^1 = \frac{3}{4} \left(\frac{2}{3} - (0.5 - 0.0417) \right) = \frac{3}{4}(0.6667 - 0.4583) = 0.156.$$

$$\mathbf{3.3} \quad (\text{a}) \quad E(X) = \int_0^2 x \cdot \frac{3}{8}x^2 \, dx = \frac{3}{8} \left[\frac{x^4}{4} \right]_0^2 = \frac{3}{8} \cdot 4 = \frac{3}{2}.$$

$$(\text{b}) \quad F(m) = \int_0^m \frac{3}{8}x^2 \, dx = \frac{m^3}{8} = 0.5 \Rightarrow m^3 = 4 \Rightarrow m = 1.59.$$