

3.5 Integration (by parts, substitution, partial fractions)

Name: _____ Class: _____ Date: _____

Total: 16 marks

KEY WORDS substitution 换元积分 • integration 积分 • integration by parts 分部积分 • partial fractions 部分分式
• rational function 有理函数

Objective

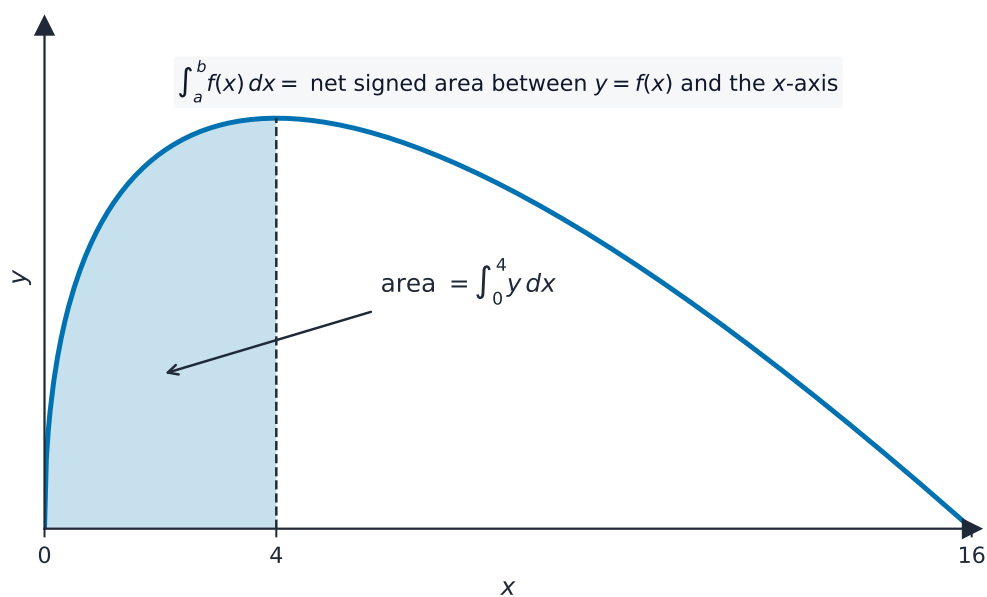
Build the skills to answer exam questions on **integration** 积分 in Pure 3 — the standard integral $\frac{1}{x^2 + a^2}$, **integration by parts** 分部积分, **substitution** 换元积分, partial fractions, and the $\frac{kf'(x)}{f(x)}$ pattern.

You must be able to:

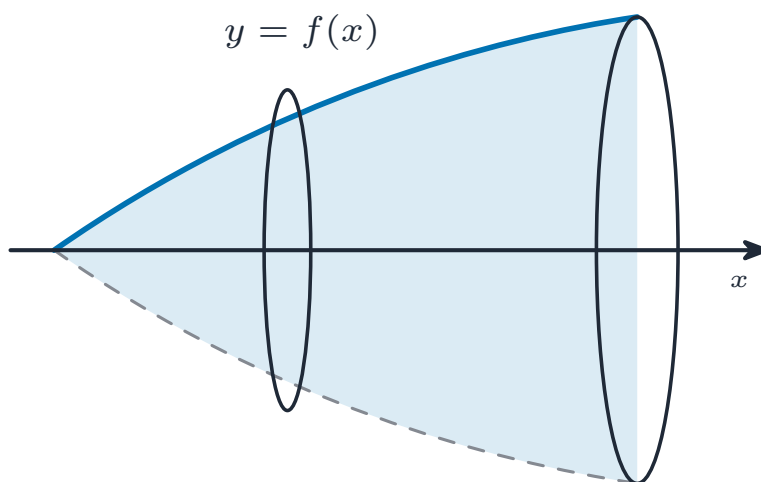
- recognise $\int \frac{1}{x^2 + a^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a} + C$ and $\int \frac{kf'(x)}{f(x)} dx = k \ln |f(x)| + C$
- integrate by parts and by a given substitution
- integrate a rational function via partial fractions

1 Worked examples

- By parts & the log pattern



A definite integral is still an area; Pure 3 adds the methods to evaluate harder ones



rotate the region around the x -axis to sweep a solid

An application of integration: a volume of revolution

$$\int x \cos x dx \text{ — by parts } (u = x, dv = \cos x dx): = x \sin x - \int \sin x dx = x \sin x + \cos x + C.$$

$$\int \frac{2x}{x^2 + 1} dx = \ln(x^2 + 1) + C \text{ (top is the derivative of the bottom).}$$

2 Practice

2.1 Find $\int \frac{1}{x^2 + 9} dx$. [2]

2.2 Find $\int \frac{3x^2}{x^3 + 1} dx$. [2]

3 Exam-style questions

3.1 $\int xe^x dx$ equals: [1]

- **A** $xe^x - e^x + C$
- **B** $xe^x + e^x + C$
- **C** $\frac{1}{2}x^2e^x + C$
- **D** $e^x + C$

3.2 Use the substitution $u = 1 + x^2$ to find $\int_0^2 x\sqrt{1 + x^2} dx$. [5]

3.3 (a) Express $\frac{4}{(x)(x + 2)}$ in partial fractions. [3]

(b) Hence find $\int_1^3 \frac{4}{x(x+2)} dx$, giving an exact answer. [3]

Solutions

2.1 $\frac{1}{3} \tan^{-1} \frac{x}{3} + C.$

2.2 $\ln |x^3 + 1| + C.$

3.1 A. By parts: $xe^x - \int e^x dx = xe^x - e^x + C.$

3.2 $u = 1 + x^2 \Rightarrow du = 2x dx$, so $x dx = \frac{1}{2} du$; limits $x = 0 \rightarrow u = 1$, $x = 2 \rightarrow u = 5$;
 $\int_1^5 \frac{1}{2} \sqrt{u} du = \frac{1}{2} \left[\frac{2}{3} u^{3/2} \right]_1^5 = \frac{1}{3} (5^{3/2} - 1); = \frac{1}{3} (11.180 - 1) = 3.39.$

3.3 (a) $4 = A(x + 2) + Bx$; $x = 0 \Rightarrow A = 2$; $x = -2 \Rightarrow B = -2$; $\frac{2}{x} - \frac{2}{x+2}.$

(b) $\int_1^3 \left(\frac{2}{x} - \frac{2}{x+2} \right) dx = \left[2 \ln x - 2 \ln(x+2) \right]_1^3 = 2 \ln \frac{x}{x+2} \Big|_1^3 = 2 \left(\ln \frac{3}{5} - \ln \frac{1}{3} \right) = 2 \ln \frac{9}{5}.$