Further Mechanics

A-Level Further Mathematics ▪ Topic 3

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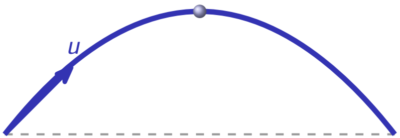


What we will cover

1. Projectiles
2. Rigid-body equilibrium
3. Circular motion
4. Hooke's law & elastic energy
5. Variable force & restitution
6. Recap

1 Projectiles and equilibrium

Projectile motion

Separate horizontal & vertical:

$$\text{range} = \frac{u^2 \sin 2\alpha}{g}, \quad H = \frac{u^2 \sin^2 \alpha}{2g}.$$

$u = 20, \ \alpha = 30^\circ$

range $= 34.6\,\mathrm{m}$, $H = 5\,\mathrm{m}$

A projectile is two independent one-dimensional motions

the model	a **projectile** 抛射体 moves freely under gravity: constant acceleration g downwards, and no horizontal acceleration
horizontal	$x = u\cos\alpha \cdot t$ – constant velocity throughout
vertical	$y = u\sin\alpha \cdot t - \frac{1}{2}gt^2$ – constant acceleration
Cartesian equation 笛卡尔方程	eliminate t: $y = x\tan\alpha - \dfrac{gx^2}{2u^2\cos^2\alpha}$ – a parabola
bounding parabola 包络抛物线	varying α at fixed u sweeps out an envelope; nothing beyond it is reachable at that speed

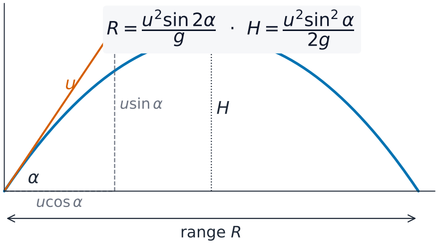
Treat the two directions separately and let *time* be the only quantity they share. Every projectile question is solved by finding t from one direction and using it in the other.

The trajectory, and the boundary of what it can reach

The trajectory	Eliminating t between the two component equations gives the Cartesian equation of the **trajectory** 轨迹 – the path – and it is a parabola.
Same speed, many angles	For one fixed launch speed, every trajectory lies under a bounding parabola: the envelope of everything reachable.
Which angle	Two launch angles reach the same range; 45° is the one that reaches furthest on level ground.

> Write x and y separately, then eliminate t. The parabola is a *result*, never the starting point.

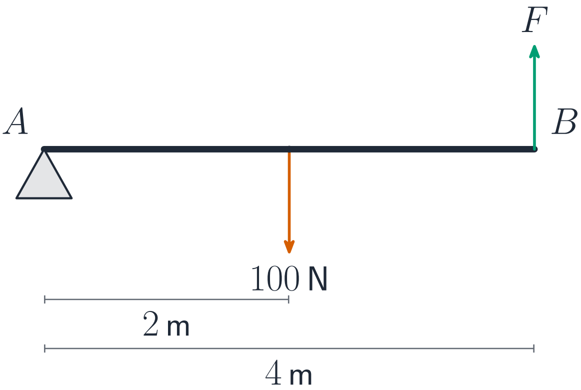
Projectile motion, in detail



$$\text{range} = \frac{u^2 \sin 2\alpha}{g}, \qquad H = \frac{u^2 \sin^2 \alpha}{2g}$$

> Treat horizontal and vertical motion separately: constant velocity across, constant acceleration down. Range is greatest at $\alpha = 45°$.

Equilibrium of a rigid body



Moment 力矩 = force × perpendicular distance; weight acts at the **centre of mass** 质心.

Equilibrium: $\sum F = 0$ and $\sum M = 0$.

Beam $4\,\mathrm{m}$, $100\,\mathrm{N}$, pivot at A

$F \times 4 = 100 \times 2 \Rightarrow F = 50\,\mathrm{N}$

Centre of mass, and the two ways a body gives way

Centre of mass	The weight acts at the **centre of mass** 质心. For a uniform flat shape – a **lamina** 薄片 – find it by **symmetry** 对称; for a composite body, treat it as a set of particles.
Toppling	**Toppling** 翻倒: the body turns over about an edge, once the line of the weight falls outside the base.
Sliding	**Sliding** 滑动: it slips instead, once the applied force exceeds the limiting friction.
Which comes first	Compare the two thresholds. The smaller one happens; an exam question usually asks which, and why.

> Equilibrium needs both conditions: the vector sum of the **coplanar forces** 共面力 is zero *and* the sum of the moments about any point is zero.

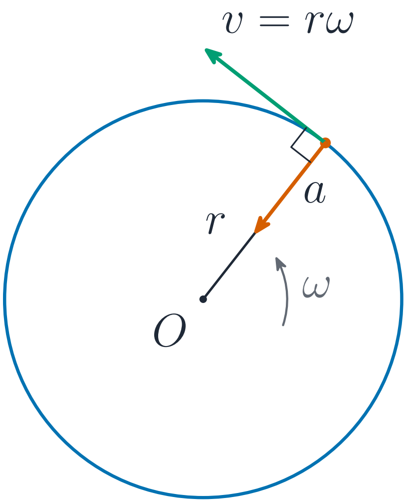
Equilibrium of a rigid body needs two conditions

condition 1	the vector sum of the forces is zero – so no translational acceleration
condition 2	the sum of the moments about any point is zero – so no rotational acceleration
coplanar forces 共面力	all in one plane, so condition 1 is two scalar equations and condition 2 is one
choosing the point	take moments about a point where an unknown force acts – it contributes no moment and drops out
three forces	if a rigid body is in equilibrium under exactly three non-parallel forces, their lines of action are concurrent

> Either condition alone is not equilibrium. A body can have zero net force and still spin – that is a couple, and it is a favourite exam trap.

2 Circular motion and elasticity

Circular motion



$v = r\omega$; the **centripetal acceleration** 向心加速度 points to the centre:

$$a = r\omega^2 = \frac{v^2}{r}.$$

$r = 2,\ \omega = 3\,\mathrm{rad/s}$
$v = 6\,\mathrm{m/s},\ a = 18\,\mathrm{m/s^2}$

The hammer thrower is a moving diagram

> The hammer thrower is a moving diagram of this page. The taut wire pulls the ball toward the centre – that inward pull is the centripetal force – while the ball's velocity points along the t

Horizontal circles, vertical circles

Angular speed	**Angular speed 角速度** ω links to the speed by $v = r\omega$, and the centripetal acceleration is $a = r\omega^2 = v^2/r$.
Horizontal circle	In a **horizontal circle 水平圆** the speed is constant – as in a **conical pendulum 圆锥摆**, a mass swung on a string that traces a cone.
Vertical circle	In a **vertical circle 竖直圆** the speed changes with height, so use energy conservation; the **normal contact force 法向接触力** or the string tension supplies the centripetal force.
Where it breaks	The string goes slack (or contact is lost) the moment the required centripetal force exceeds what tension or contact can provide.

Constant speed $\Rightarrow$ Newton's second law alone. Changing speed $\Rightarrow$ energy first, then Newton's second law at the point asked about.

Hooke's law and elastic energy

$$T = \frac{\lambda x}{L}, \qquad E = \frac{\lambda x^2}{2L}.$$

λ is the **modulus of elasticity** 弹性模量.

$L = 2$, $\lambda = 50$, $x = 0.5\,\mathrm{m}$

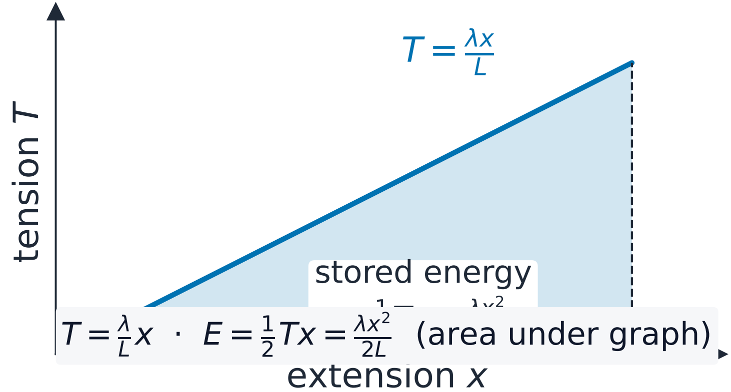
$T = 12.5\,\mathrm{N}$, $E = 3.125\,\mathrm{J}$

Elastic potential energy

The stored energy	Stretching stores **elastic potential energy** 弹性势能 $E = \dfrac{\lambda x^2}{2L}$, where L is the natural length and λ the **modulus of elasticity** 弹性模量.
Why the square	The force grows with the extension, so the energy is the *area* under the force–extension line – hence x^2, not x.
In an energy equation	It joins kinetic and gravitational potential energy in a conservation statement; the string doing negative work *is* the energy it stores.

> A stretched string is a battery: whatever went in comes back out when it returns to its natural length.

Hooke's law and elastic energy, in detail



$T = \frac{\lambda}{L}x$ · $E = \frac{1}{2}Tx = \frac{\lambda x^2}{2L}$ (area under graph)

$L = 2,\ \lambda = 50,\ x = 0.5\,\mathrm{m}$ $T = 12.5\,\mathrm{N}$, $E = 3.125\,\mathrm{J}$

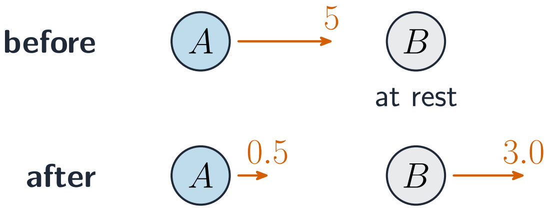
3 Variable force and momentum

Variable force and momentum

A position-dependent **variable force** 变力 uses

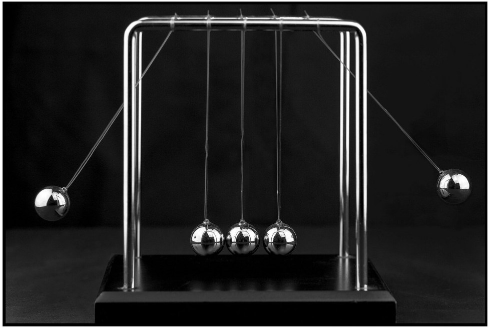
$$a = v\frac{dv}{dx},$$

turning Newton's law into a differential equation in v and x.



$$e = \frac{\text{speed of separation}}{\text{speed of approach}} = \frac{3.0 - 0.5}{5} = 0.5$$

Momentum



A Newton's cradle demonstrates conservation of momentum in collisions.

Worked example – a collision with restitution

Sphere A (2 kg, 5 m s^{-1}) hits stationary B (3 kg), with $e = 0.5$. Find both final speeds.

- **momentum** 动量: $2(5) = 2v_A + 3v_B$, so $10 = 2v_A + 3v_B$
- **restitution** 恢复系数: $v_B - v_A = e(5 - 0) = 2.5$
- substituting: $10 = 2v_A + 3(v_A + 2.5)$, so $v_A = 0.5$ m s^{-1}
- $v_B = 3.0$ m s^{-1}

Two equations, always the same two: **conservation of linear momentum** 动量守恒 and Newton's law of restitution,

$$e = \frac{\text{separation speed}}{\text{approach speed}}.$$

Check $v_B > v_A$, or the spheres would still be colliding.

How bouncy is the collision?

Newton's experimental law	**Newton's experimental law** 牛顿实验定律 defines the **coefficient of restitution** 恢复系数: $e = \dfrac{\text{speed of separation}}{\text{speed of approach}}$, with $0 \leqslant e \leqslant 1$.
The two ends	$e = 1$ is perfectly elastic – no kinetic energy lost. $e = 0$ is inelastic: the bodies stay together.
Oblique impact	In an **oblique impact** 斜碰撞, resolve the velocities along and perpendicular to the line of impact. Restitution applies along it; the perpendicular component is unchanged.

Momentum is conserved in *every* collision; restitution is the second equation you need, and only it tells you about the energy.

Linear motion under a variable force

Newton's law

$$8v\frac{dv}{dx} = x^3 + 4x$$

$$\int 8v\,dv = \int (x^3+4x)\,dx$$

separate and integrate

$$4v^2 = \frac{1}{4}x^4 + 2x^2 + C$$

apply $x = 0,\ v = 1$: $C = 4$, so $v = \frac{1}{4}x^2 + 1$

a position-dependent force becomes a **differential equation** in v and x

> A position-dependent force becomes a differential equation in v and x

Use dv/dt when the force depends on time

which form of acceleration?

force depends on **time** t	force depends on **position** x
$a = \dfrac{dv}{dt}$	$a = v\,\dfrac{dv}{dx}$
gives v as a function of t	gives v as a function of x

both come from $a = \dfrac{dv}{dt}$ with the chain rule; pick the one matching what the force depends on

> Use dv/dt when the force depends on time, v dv/dx when it depends on position

Exam tips

- For **projectiles**, resolve into horizontal (constant velocity) and vertical ($a = g$) motion, linked by the same time.
- For a **rigid body in equilibrium**, take moments about a point that removes an unknown force.
- For **circular motion**, use $F = mv^2/r = m\omega^2 r$ towards the centre; in a vertical circle check the minimum speed at the top.
- With a **variable force**, use $a = v\frac{dv}{dx}$ and integrate; elastic PE $= \frac{\lambda x^2}{2L}$.

4 Recap

Topic 3 – key takeaways

1 Projectiles

range $u^2 \sin 2\alpha / g$; parabolic path

2 Circular

$v = r\omega$, $a = v^2/r$ toward the centre

3 Hooke

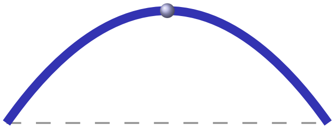
$T = \lambda x/L$; $E = \lambda x^2/2L$

4 Restitution

$e =$ separation/approach speed

Check yourself

1. Direction of the centripetal acceleration?
2. Tension formula for an elastic string?
3. What is e for a perfectly elastic collision?
4. Which form of a suits a force depending on x?



Answers **1.** toward the centre **2.** $T = \lambda x / L$ **3.** $e = 1$ **4.** $a = v\,dv/dx$