Further Pure Mathematics 2

A-Level Further Mathematics ▪ Topic 2

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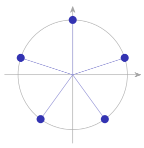
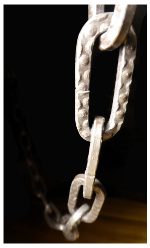


What we will cover



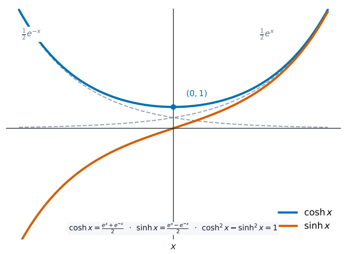
1 Hyperbolic and matrices

Hyperbolic functions

$$\sinh x = \frac{e^x - e^{-x}}{2}, \quad \cosh x = \frac{e^x + e^{-x}}{2}, \quad \tanh x = \frac{\sinh x}{\cosh x}$$

Identities echo trig, with sign changes: $\cosh^2 x - \sinh^2 x = 1$. $(\sinh x)' = \cosh x$, $(\cosh x)' = \sinh x$.

Hyperbolic functions, in detail



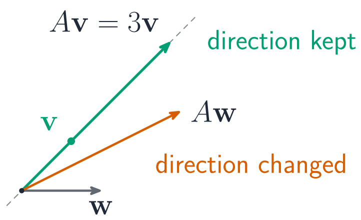
$\cosh$ is the average of e^x and e^{-x}; $\sinh$ is half their difference.

The hyperbolic identities, and the logarithmic forms

the main identity	$\cosh^2 x - \sinh^2 x = 1$ – note the minus, where the trig version has a plus
the reciprocals	$\operatorname{sech} x = 1/\cosh x$, $\operatorname{cosech} x = 1/\sinh x$, $\coth x = 1/\tanh x$
derived	$1 - \tanh^2 x = \operatorname{sech}^2 x$, and $\coth^2 x - 1 = \operatorname{cosech}^2 x$
logarithmic form 对数形式	$\sinh^{-1} x = \ln\left(x + \sqrt{x^2+1}\right)$, $\cosh^{-1} x = \ln\left(x + \sqrt{x^2-1}\right)$ for $x \geq 1$
and	$\tanh^{-1} x = \frac{1}{2}\ln\frac{1+x}{1-x}$ for $\lvert x\rvert < 1$

Osborn's rule: take a trig identity and flip the sign of every product of two sines. That generates the hyperbolic identity you need.

Eigenvalues and eigenvectors



Characteristic equation 特征方程 $\det(A - \lambda I) = 0$ gives the **eigenvalues** 特征值; $A\mathbf{v} = \lambda\mathbf{v}$ the **eigenvectors** 特征向量.

$A = \left(\begin{smallmatrix} 2 & 1 \\ 1 & 2 \end{smallmatrix}\right)$

$(2 - \lambda)^2 - 1 = 0 \Rightarrow \lambda = 1, 3$

Then $A = QDQ^{-1}$.

Solving three equations as one matrix equation

the matrix equation 矩阵方程	three linear equations in three unknowns become $A\mathbf{x} = \mathbf{b}$
non-singular A	$\det A \neq 0$, so $\mathbf{x} = A^{-1}\mathbf{b}$ – exactly one solution
singular A	$\det A = 0$: either no solution, or infinitely many. The geometry is three planes with no common point, or a common line
diagonalisation 对角化	$A = PDP^{-1}$ with D **diagonal** 对角矩阵 of eigenvalues and P their eigenvectors
why it helps	$A^n = PD^nP^{-1}$, and raising a diagonal matrix to a power is done entry by entry

> Check $\det A$ first. It decides whether you invert, or whether you must describe the family of solutions instead.

Worked example – finding eigenvalues

Find the eigenvalues of $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$.

- solve $\det(A - \lambda I) = 0$
- $\begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = (2-\lambda)^2 - 1 = 0$
- $\lambda^2 - 4\lambda + 3 = 0$, so $\lambda = 1$ or 3

> Two checks, both free: the eigenvalues sum to the trace $(1 + 3 = 4)$ and multiply to the determinant $(1 \times 3 = 3)$. Then substitute each λ back to get its eigenvector.

2 Calculus

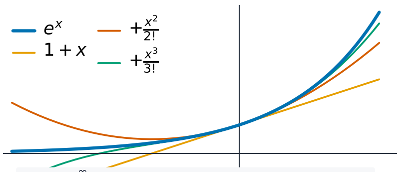
Differentiation and Maclaurin series

Maclaurin series 麦克劳林级数:

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots$$

e.g. $e^x = 1 + x + \frac{x^2}{2!} + \cdots$ – more terms, wider match.

Differentiation and Maclaurin series, in detail



$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n \cdot e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

> Adding more terms makes the polynomial match e^x over a wider stretch around $x = 0$.

Integration

Standard integrals:

$$\int \frac{dx}{a^2 + x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a} + C,$$

$$\int \frac{dx}{\sqrt{a^2 - x^2}} = \sin^{-1}\frac{x}{a} + C.$$

$\int \frac{dx}{\sqrt{4-x^2}}$

$= \sin^{-1}\frac{x}{2} + C$

A **reduction formula** 递推公式 links I_n to I_{n-1}.

Bounding a sum by an integral

the picture	draw rectangles of unit width under, and over, a decreasing curve such as $y = 1/x$
each rectangle	has height $\frac{1}{r}$ and width 1, so its area *is* the term $\frac{1}{r}$
under-estimate	rectangles below the curve give $\sum_{r=2}^{n}\frac{1}{r} < \int_1^n \frac{dx}{x}$
over-estimate	rectangles above it give $\int_1^{n+1}\frac{dx}{x} < \sum_{r=1}^{n}\frac{1}{r}$
the conclusion	the sum sits between two logarithms, so the harmonic series diverges – slowly, like $\ln n$

Sketch the rectangles before writing any inequality. Which side of the curve they sit on decides the direction of the sign, and that is the whole method.

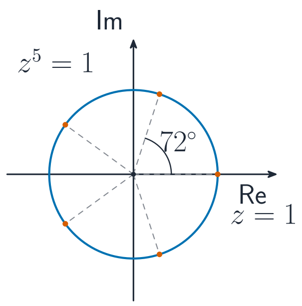
The standard integrals to know by sight

inverse tan	$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + C$
inverse sin	$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a} + C$
inverse sinh	$\int \frac{1}{\sqrt{a^2+x^2}}\,dx = \sinh^{-1}\frac{x}{a} + C$
inverse cosh	$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \cosh^{-1}\frac{x}{a} + C$
arc length 弧长	$\int \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx$, and $\int \sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta$ in polars

The sign inside the surd chooses the family: $a^2 - x^2$ gives a trig inverse, $a^2 + x^2$ or $x^2 - a^2$ a hyperbolic one. Complete the square first when the quadratic is not already in that shape.

3 Complex and DEs

Complex numbers – de Moivre



De Moivre 棣莫弗定理:

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta.$$

Gives multiple-angle formulae and the n **roots of unity** 单位根 (equally spaced).

Four techniques from the calculus half

Maclaurin's series	**Maclaurin's series** 麦克劳林级数: $f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots$ – a Taylor series about the origin.
Separable variables	A differential equation with **separable variables** 可分离变量 is integrated after gathering each variable on its own side; one condition fixes the **particular solution** 特解.
Riemann sum	A **Riemann sum** 黎曼和 bounds a sum by an integral – rectangles above and below the curve give the two bounds.
Surface of revolution	The **surface area of revolution** 旋转曲面面积 is $2\pi\int y\sqrt{1+(dy/dx)^2}\,dx$ – the arc length element carried round the axis.

Completing the square 配方 is the step that turns an unfamiliar integrand into a standard arctan or arcsin form.

Differential equations

First-order linear: multiply by the **integrating factor** 积分因子 $\mu = e^{\int P\,dx}$.

$\frac{dy}{dx} + 2y = e^x$

$\mu = e^{2x} \Rightarrow y = \frac{1}{3}e^x + Ce^{-2x}$

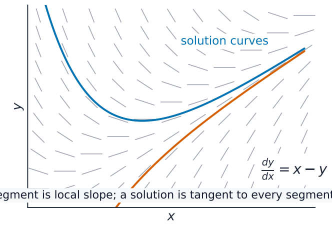
Constant-coefficient: **complementary function** 余函数 + **particular integral** 特积分.

The structure of a linear differential equation's solution

the general solution 通解	is the sum of two parts
complementary function 余函数	the solution of the equation with the right side set to zero
particular integral 特解	any single solution of the full equation – guessed from the form of the right side
auxiliary equation 辅助方程	replace $\frac{d^2y}{dx^2} \to m^2$, $\frac{dy}{dx} \to m$, $y \to 1$; its roots give the complementary function
the three cases	distinct real roots $\to Ae^{m_1x} + Be^{m_2x}$; a repeated root $\to (A+Bx)e^{mx}$; complex $\to e^{px}(A\cos qx + B\sin qx)$

Find the complementary function first, then the particular integral, then apply the boundary conditions to the sum – never to one part alone.

Differential equations, in detail



each segment is local slope; a solution is tangent to every segment it meets

First-order linear: multiply by the **integrating factor** 积分因子 $\mu = e^{\int P\,dx}$.

Worked example – an integrating factor, and a repeated root

(a) Solve $\frac{dy}{dx} + 2y = e^x$. **(b) Solve** $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 0$.

- (a) integrating factor $e^{\int 2\,dx} = e^{2x}$; so $\frac{d}{dx}(ye^{2x}) = e^{3x}$
- $ye^{2x} = \frac{1}{3}e^{3x} + C$, giving $y = \frac{1}{3}e^x + Ce^{-2x}$
- (b) auxiliary: $m^2 - 4m + 4 = (m-2)^2 = 0$, a repeated root $m = 2$
- $y = (A + Bx)e^{2x}$

A repeated root needs the extra factor of x – without it the two terms are the same function and you have only one arbitrary constant for a second-order equation.

Complex numbers

Self-similar patterns like Romanesco broccoli arise from iterating functions in the complex plane.

Exam tips

- Learn the **hyperbolic identities** and derivatives, and use the inverse hyperbolic functions in integration.
- Solve a linear **differential equation** as complementary function + particular integral, then apply the boundary conditions.
- Use **de Moivre's theorem** for powers and roots of complex numbers and to derive trigonometric identities.
- Reduce an integral to a standard form by a stated substitution.

4 Recap

Topic 2 – key takeaways

1 Hyperbolic
$\cosh^2 - \sinh^2 = 1$

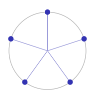
2 Eigenvalues
$\det(A - \lambda I) = 0$; $A = QDQ^{-1}$

3 Complex
de Moivre; roots of unity

4 DEs
integrating factor; CF + PI

Check yourself

1. Simplify $\cosh^2 x - \sinh^2 x$.
2. What equation gives the eigenvalues?
3. State de Moivre's theorem.
4. What multiplies a first-order linear DE?



Answers **1.** 1 **2.** $\det(A - \lambda I) = 0$ **3.** $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$ **4.** the integrating factor