

Further Pure Mathematics 1

A-Level Further Mathematics • Topic 1

A-Level Further Mathematics



Further Pure Mathematics 1

What we will cover

- 1 Roots & series
- 2 Rational functions
- 3 Matrices
- 4 Polar coordinates
- 5 Vectors & induction
- 6 Recap



1 Roots and series

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└─ Roots and series

Roots and summation of series

Roots $\leftrightarrow$ coefficients (quadratic):

$$\alpha + \beta = -\frac{b}{a}, \quad \alpha\beta = \frac{c}{a}.$$

Standard sums: $\sum r = \frac{1}{2}n(n+1)$,
 $\sum r^2 = \frac{1}{6}n(n+1)(2n+1)$.

$$\begin{aligned} \sum_{r=1}^n (2r+1) \\ = 2 \cdot \frac{1}{2}n(n+1) + n = n(n+2) \end{aligned}$$

Method of differences 差分法 cancels middle terms.

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└─ Roots and series

Roots and coefficients, without solving anything

quadratic for $ax^2 + bx + c = 0$ with roots α, β : $\alpha + \beta = -\frac{b}{a}$, $\alpha\beta = \frac{c}{a}$

cubic for $ax^3 + bx^2 + cx + d = 0$: $\sum \alpha = -\frac{b}{a}$, $\sum \alpha\beta = \frac{c}{a}$, $\alpha\beta\gamma = -\frac{d}{a}$

the useful identity $\alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta$ – and the cubic analogue $\sum \alpha^2 = (\sum \alpha)^2 - 2\sum \alpha\beta$

discriminant 判别式 $b^2 - 4ac$ decides how many real roots there are before you find any of them

the point symmetric functions of the roots come straight from the **coefficients** 系数 – the roots themselves are never needed

When a question asks for $\alpha^2 + \beta^2$ or $\frac{1}{\alpha} + \frac{1}{\beta}$, do not solve the equation. Build it from the two sums.

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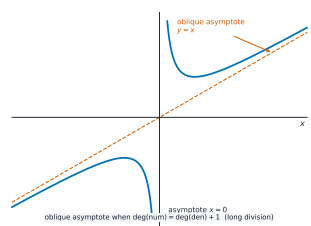
└─ Roots and series

Rational functions

Vertical asymptotes: denominator zero (numerator not). Horizontal: compare degrees.

If the numerator's degree is one more than the denominator, divide first – the quotient is an **oblique** asymptote.

Rational functions, in detail



Far from the origin the curve hugs the slant line $y = x$; near $x = 0$ it runs off along the vertic

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└ Roots and series

Sketching a rational function

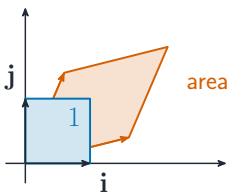
rational function 有理函数	a fraction of two polynomials
vertical asymptotes	where the denominator is zero (and the numerator is not)
horizontal	when the degrees are equal, y tends to the ratio of the leading coefficients
oblique asymptote 斜渐近线	when the top's degree is exactly one higher : divide out to get $y = mx + c$ plus a vanishing remainder
the checks	axis intercepts, sign of the function between asymptotes, and any stationary points

The oblique case is the one the exam sets, and **polynomial division** is how you find it. Divide first, sketch second.

2 Matrices

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└ Matrices

Matrices



$\det A = ad - bc$ (the area scale factor).

$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

A 2×2 matrix is a **transformation** 几何变换; AB means "do B then A ".

The matrix operations, and the transformations they are

addition	entry by entry, in matching positions – so the shapes must match
multiplication	row into column; AB needs A 's columns to equal B 's rows, and $AB \neq BA$ in general
identity matrix 单位矩阵	I , with ones on the diagonal: $AI = IA = A$
as a geometric transformation 几何变换	a 2×2 matrix maps the plane – rotation, reflection, enlargement 放大, shear, or a composition
the determinant	$\det A$ is the area scale factor; $\det A = 0$ means the transformation collapses the plane onto a line
invariant 不变 line	a line the transformation maps onto itself – found from $Av = \lambda v$

Composing transformations is multiplying matrices, **right to left**: BA means do A first. Getting that order wrong is the standard lost mark.

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└ Matrices

Matrices: the special ones, and what they do to a plane

Zero matrix	The zero matrix 零矩阵 sends every vector to the origin – and multiplying by it destroys all information.
Inverse matrix	The inverse matrix 逆矩阵 undoes the transformation: $AA^{-1} = I$, and it exists only when $\det A \neq 0$.
Singular matrix	A singular matrix 奇异矩阵 has zero determinant: it collapses the plane onto a line, so it cannot be undone.
Invariant points	Invariant points 不变点 satisfy $Ax = x$ – fixed by the transformation. A stretch 伸缩 fixes the axis it stretches along.

The determinant is the area scale factor. Zero means the area is destroyed, which is exactly why the inverse fails.

3 Polar and vectors

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Polar and vectors

Polar coordinates

Polar 极坐标 (r, θ) : $x = r \cos \theta$, $y = r \sin \theta$, $r^2 = x^2 + y^2$.

$$\text{area} = \frac{1}{2} \int r^2 d\theta.$$

e.g. $r = 4 \cos \theta \Rightarrow (x - 2)^2 + y^2 = 4$.

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Polar and vectors

Worked example – polar to Cartesian

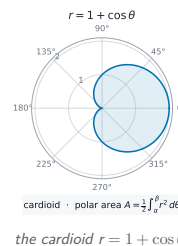
Convert $r = 4 \cos \theta$ to Cartesian coordinates 直角坐标 and identify the curve.

- multiply both sides by r : $r^2 = 4r \cos \theta$
- use $r^2 = x^2 + y^2$ and $r \cos \theta = x$:
 $x^2 + y^2 = 4x$
- complete the square: $(x - 2)^2 + y^2 = 4$
- a circle of radius 2 centred at $(2, 0)$

Multiplying by r is the move. It turns $r \cos \theta$ into x and r^2 into $x^2 + y^2$, and nothing else does.

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Polar and vectors

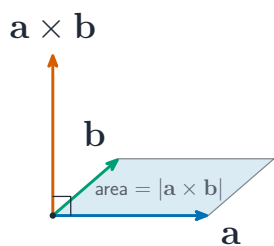
Polar coordinates, in detail



The cardioid $r = 1 + \cos \theta$, plotted straight from r as a function of θ .

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Polar and vectors

Vectors



A **plane** 平面: $r \cdot n = p$. The **vector product** 向量积

$$a \times b = |a||b| \sin \theta \hat{n}$$

is perpendicular to both; its length is the parallelogram's area.

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Polar and vectors

Three ways to write a plane

Cartesian $ax + by + cz = d$, where (a, b, c) is a normal to the plane

scalar product $r \cdot n = p$ – the same statement, with the normal made explicit

parametric $r = a + \lambda b + \mu c$: a point plus two independent directions in the plane

converting the cross product $b \times c$ gives the normal, which turns the parametric form into the other two

Choose the form the question wants: **Cartesian** for a distance from a point, **parametric** for an intersection with a line.

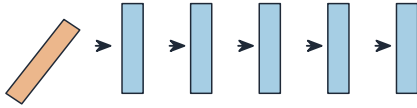
Proof by induction

Induction 数学归纳法:

- 1 **base case** $n = 1$
- 2 **inductive step**: true for $k \Rightarrow$ true for $k + 1$

Then it holds for every n .

so every domino falls — true for all n



Induction, series, and the vocabulary around them

Mathematical induction

Mathematical induction 数学归纳法: prove the base case, assume for $n = k$, then prove for $n = k + 1$. An **inductive proof** 归纳证明 must state all three steps.

Sum to infinity

A geometric series is **convergent** 收敛的 when $|r| < 1$, and its **sum to infinity** 无穷和 is $\frac{a}{1-r}$.

Partial fractions

Partial fractions 部分分式 split a rational function into simpler pieces – which is what makes a telescoping sum or an integral doable.

Substitution and curves

A **substitution** 换元 converts an integral into a standard form; **polar curves** 极坐标曲线 $r = f(\theta)$ have area $\frac{1}{2} \int r^2 d\theta$; **skew lines** 异面直线 in 3D are neither parallel nor intersecting.

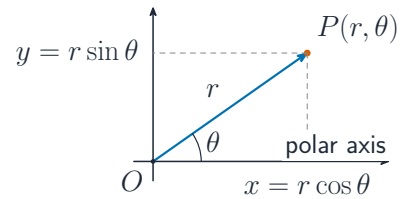
The induction step must *use* the assumption. A proof that reaches $n = k + 1$ without it has proved nothing.

Induction, and the sentence each step must contain

- | | |
|-----------------------|---|
| base case | show the statement holds for the first n – usually $n = 1$ |
| inductive step | assume it holds for $n = k$, then prove it for $n = k + 1$ |
| the conclusion | "true for $n = 1$, and true for $k + 1$ whenever true for k , so true for all $n \geq 1$ by induction" |
| conjecture 猜想 | often you compute a few cases first, guess the pattern, and only then confirm it by induction 数学归纳法 |
| where marks go | stating the assumption explicitly, and writing the conclusion sentence in full |

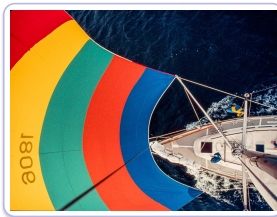
The proof is not finished when the algebra closes. The **final sentence** is a marked step, and omitting it is the commonest way to lose a mark on an otherwise correct proof.

Polar coordinates, continued



Polar coordinates (r, θ) convert to Cartesian by $x = r \cos \theta$ and $y = r \sin \theta$.

Vectors, continued



Forces like wind and water are vectors – they have both size and direction.

Worked example – a cross product

Find $\mathbf{a} \times \mathbf{b}$ for $\mathbf{a} = \mathbf{i} + \mathbf{j}$ and $\mathbf{b} = \mathbf{j} + \mathbf{k}$.

$$\begin{aligned} \mathbf{a} \times \mathbf{b} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} \\ &= \mathbf{i}(1 \cdot 1 - 0 \cdot 1) - \mathbf{j}(1 \cdot 1 - 0 \cdot 0) + \mathbf{k}(1 \cdot 1 - 1 \cdot 0) \\ &= \mathbf{i} - \mathbf{j} + \mathbf{k} \end{aligned}$$

Check it: the result is **perpendicular to both**, so $\mathbf{a} \cdot (\mathbf{a} \times \mathbf{b})$ must be zero. It is. That check costs seconds and catches every sign slip.

Exam tips

- Use the **relationships between roots and coefficients** (sum, sum of pairs, product) instead of solving the polynomial.
- For **proof by induction**, set out the base case, assume the result for $n = k$, prove it for $n = k + 1$, and write a clear closing statement.
- Sketch **rational-function graphs** with their asymptotes and turning points; use the **method of differences** for a summation.
- For **matrices**, know the determinant, inverse and invariant lines, and interpret each transformation geometrically.

4

Recap

Topic 1 – key takeaways

1 Roots

sums & products link to coefficients

2 Matrices

$\det = ad - bc$; transformations

3 Polar

$x = r \cos \theta$; area $\frac{1}{2} \int r^2 d\theta$

4 Induction

base case + inductive step

Check yourself

- 1 For $ax^2 + bx + c = 0$, what is $\alpha\beta$?
- 2 What is $\det \begin{pmatrix} 3 & 1 \\ 2 & 4 \end{pmatrix}$?
- 3 Convert (r, θ) to the x -coordinate.
- 4 The two steps of proof by induction?



Answers 1. c/a 2. 10 3. $x = r \cos \theta$ 4. base case, then inductive step