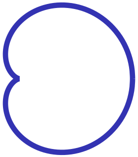
Further Pure Mathematics 1

A-Level Further Mathematics ▪ Topic 1

A-Level Further Mathematics



What we will cover

1. Roots & series
2. Rational functions
3. Matrices
4. Polar coordinates
5. Vectors & induction
6. Recap

1 Roots and series

Roots and summation of series

Roots $\leftrightarrow$ coefficients (quadratic):

$$\alpha + \beta = -\frac{b}{a}, \quad \alpha\beta = \frac{c}{a}.$$

Standard sums: $\sum r = \frac{1}{2}n(n+1)$, $\sum r^2 = \frac{1}{6}n(n+1)(2n+1)$.

$\sum_{r=1}^{n}(2r+1)$

$= 2 \cdot \frac{1}{2}n(n+1) + n = n(n+2)$

Method of differences 差分法 cancels middle terms.

Roots and coefficients, without solving anything

quadratic	for $ax^2 + bx + c = 0$ with roots α, β: $\alpha + \beta = -\frac{b}{a}$, $\alpha\beta = \frac{c}{a}$
cubic	for $ax^3 + bx^2 + cx + d = 0$: $\sum \alpha = -\frac{b}{a}$, $\sum \alpha\beta = \frac{c}{a}$, $\alpha\beta\gamma = -\frac{d}{a}$
the useful identity	$\alpha^2+\beta^2 = (\alpha+\beta)^2 - 2\alpha\beta$ – and the cubic analogue $\sum \alpha^2 = (\sum \alpha)^2 - 2\sum \alpha\beta$
discriminant 判别式	$b^2 - 4ac$ decides how many real roots there are before you find any of them
the point	symmetric functions of the roots come straight from the **coefficients** 系数 – the roots themselves are never needed

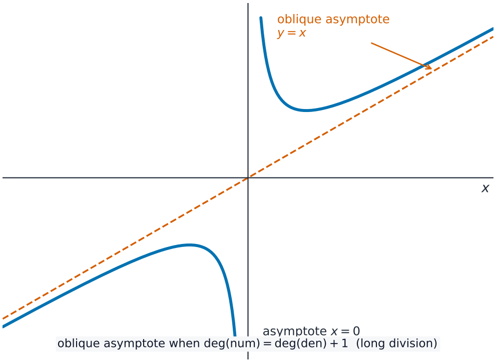
When a question asks for $\alpha^2 + \beta^2$ or $\frac{1}{\alpha} + \frac{1}{\beta}$, do not solve the equation. Build it from the two sums.

Rational functions

Vertical asymptotes: denominator zero (numerator not). Horizontal: compare degrees.

> If the numerator's degree is one more than the denominator, divide first – the quotient is an oblique asymptote.

Rational functions, in detail



Far from the origin the curve hugs the slant line $y = x$; near $x = 0$ it runs off along the vertic

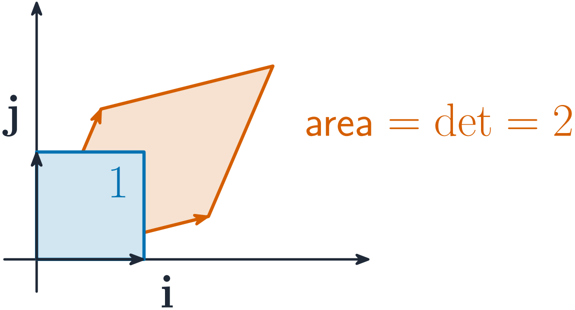
Sketching a rational function

rational function 有理函数	a fraction of two polynomials
vertical asymptotes	where the denominator is zero (and the numerator is not)
horizontal	when the degrees are equal, y tends to the ratio of the leading coefficients
oblique asymptote 斜渐近线	when the top's degree is exactly one higher: divide out to get $y = mx + c$ plus a vanishing remainder
the checks	axis intercepts, sign of the function between asymptotes, and any stationary points

> The oblique case is the one the exam sets, and polynomial division is how you find it. Divide first, sketch second.

2 Matrices

Matrices



$\det A = ad - bc$ (the area scale factor).

$$A^{-1} = \frac{1}{ad - bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

A 2×2 matrix is a **transformation** 几何变换; AB means "do B then A".

The matrix operations, and the transformations they are

addition	entry by entry, in matching positions – so the shapes must match
multiplication	row into column; AB needs A's columns to equal B's rows, and $AB \neq BA$ in general
identity matrix 单位矩阵	I, with ones on the diagonal: $AI = IA = A$
as a geometric transformation 几何变换	a 2×2 matrix maps the plane – rotation, reflection, **enlargement 放大**, shear, or a composition
the determinant	$\det A$ is the area scale factor; $\det A = 0$ means the transformation collapses the plane onto a line
invariant 不变 line	a line the transformation maps onto itself – found from $A\mathbf{v} = \lambda\mathbf{v}$

Composing transformations is multiplying matrices, right to left: BA means do A first. Getting that order wrong is the standard lost mark.

Matrices: the special ones, and what they do to a plane

Zero matrix	The **zero matrix** 零矩阵 sends every vector to the origin – and multiplying by it destroys all information.
Inverse matrix	The **inverse matrix** 逆矩阵 undoes the transformation: $\mathbf{A}\mathbf{A}^{-1} = \mathbf{I}$, and it exists only when $\det \mathbf{A} \neq 0$.
Singular matrix	A **singular matrix** 奇异矩阵 has zero determinant: it collapses the plane onto a line, so it cannot be undone.
Invariant points	**Invariant points** 不变点 satisfy $\mathbf{A}\mathbf{x} = \mathbf{x}$ – fixed by the transformation. A **stretch** 伸缩 fixes the axis it stretches along.

> The determinant is the area scale factor. Zero means the area is destroyed, which is exactly why the inverse fails.

3 Polar and vectors

Polar coordinates

Polar 极坐标 (r, θ): $x = r\cos\theta$, $y = r\sin\theta$, $r^2 = x^2 + y^2$.

$$\text{area} = \frac{1}{2}\int r^2\, d\theta.$$

e.g. $r = 4\cos\theta \Rightarrow (x-2)^2 + y^2 = 4$.

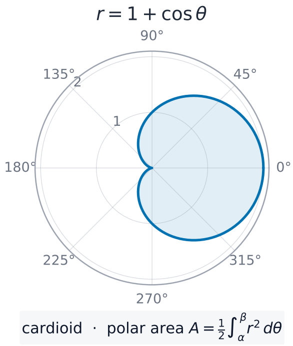
Worked example – polar to Cartesian

Convert $r = 4\cos\theta$ to Cartesian coordinates 直角坐标 and identify the curve.

- multiply both sides by r: $r^2 = 4r\cos\theta$
- use $r^2 = x^2 + y^2$ and $r\cos\theta = x$:
  $x^2 + y^2 = 4x$
- complete the square: $(x-2)^2 + y^2 = 4$
- a circle of radius 2 centred at $(2, 0)$

Multiplying by r is the move. It turns $r\cos\theta$ into x and r^2 into $x^2 + y^2$, and nothing else does.

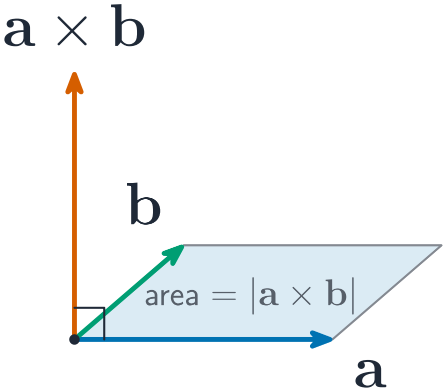
Polar coordinates, in detail



the cardioid $r = 1 + \cos\theta$

> The cardioid $r = 1 + \cos\theta$, plotted straight from r as a function of θ.

Vectors



A **plane** 平面: $\mathbf{r} \cdot \mathbf{n} = p$. The **vector product** 向量积

$$\mathbf{a} \times \mathbf{b} = |\mathbf{a}||\mathbf{b}| \sin\theta \, \hat{\mathbf{n}}$$

is perpendicular to both; its length is the parallelogram's area.

Three ways to write a plane

Cartesian	$ax + by + cz = d$, where (a, b, c) is a normal to the plane
scalar product	$\mathbf{r} \cdot \mathbf{n} = p$ – the same statement, with the normal made explicit
parametric	$\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$: a point plus two independent directions in the plane
converting	the cross product $\mathbf{b} \times \mathbf{c}$ gives the normal, which turns the parametric form into the other two

> Choose the form the question wants: Cartesian for a distance from a point, parametric for an intersection with a line.

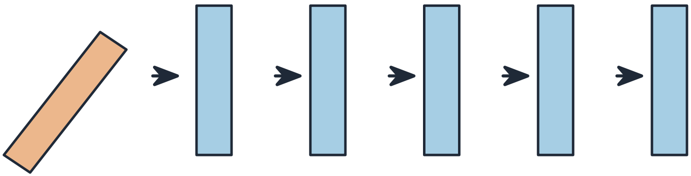
Proof by induction

Induction 数学归纳法:

1. base case $n = 1$
2. inductive step: true for $k \Rightarrow$ true for $k + 1$

Then it holds for every n.

so every domino falls — true for all n



Induction, series, and the vocabulary around them

Mathematical induction	**Mathematical induction 数学归纳法**: prove the base case, assume for $n = k$, then prove for $n = k + 1$. An **inductive proof 归纳证明** must state all three steps.
Sum to infinity	A geometric series is **convergent 收敛的** when $\lvert r \rvert < 1$, and its **sum to infinity 无穷和** is $\dfrac{a}{1-r}$.
Partial fractions	**Partial fractions 部分分式** split a rational function into simpler pieces – which is what makes a telescoping sum or an integral doable.
Substitution and curves	A **substitution 换元** converts an integral into a standard form; **polar curves 极坐标曲线** $r = f(\theta)$ have area $\frac{1}{2}\int r^2\,d\theta$; **skew lines 异面直线** in 3D are neither parallel nor intersecting.

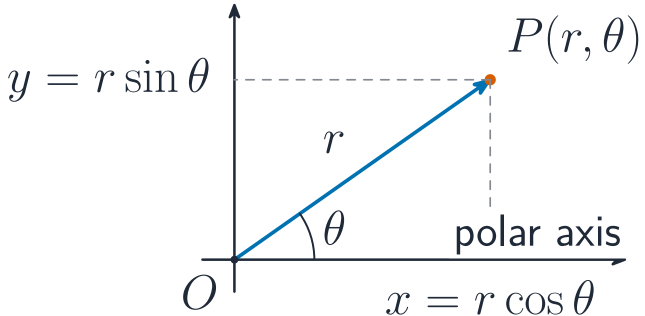
The induction step must *use* the assumption. A proof that reaches $n = k + 1$ without it has proved nothing.

Induction, and the sentence each step must contain

base case	show the statement holds for the first n – usually $n = 1$
inductive step	assume it holds for $n = k$, then prove it for $n = k + 1$
the conclusion	"true for $n = 1$, and true for $k + 1$ whenever true for k, so true for all $n \geq 1$ by induction"
conjecture 猜想	often you compute a few cases first, guess the pattern, and only then confirm it by **induction 数学归纳法**
where marks go	stating the assumption explicitly, and writing the conclusion sentence in full

The proof is not finished when the algebra closes. The final sentence is a marked step, and omitting it is the commonest way to lose a mark on an otherwise correct proof.

Polar coordinates, continued



Polar coordinates (r,θ) convert to Cartesian by $x = r\cos\theta$ and $y = r\sin\theta$.

Vectors, continued

Forces like wind and water are vectors – they have both size and direction.

Worked example – a cross product

Find $\mathbf{a} \times \mathbf{b}$ for $\mathbf{a} = \mathbf{i} + \mathbf{j}$ and $\mathbf{b} = \mathbf{j} + \mathbf{k}$.

- $\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix}$
- $= \mathbf{i}(1\cdot1 - 0\cdot1) - \mathbf{j}(1\cdot1 - 0\cdot0) + \mathbf{k}(1\cdot1 - 1\cdot0)$
- $= \mathbf{i} - \mathbf{j} + \mathbf{k}$

Check it: the result is perpendicular to both, so $\mathbf{a} \cdot (\mathbf{a} \times \mathbf{b})$ must be zero. It is. That check costs seconds and catches every sign slip.

Exam tips

- Use the **relationships between roots and coefficients** (sum, sum of pairs, product) instead of solving the polynomial.
- For **proof by induction**, set out the base case, assume the result for $n = k$, prove it for $n = k + 1$, and write a clear closing statement.
- Sketch **rational-function graphs** with their asymptotes and turning points; use the **method of differences** for a summation.
- For **matrices**, know the determinant, inverse and invariant lines, and interpret each transformation geometrically.

4 Recap

Topic 1 – key takeaways

1 Roots

sums & products link to coefficients

2 Matrices

$\det = ad - bc$; transformations

3 Polar

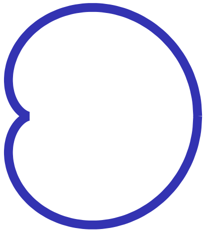
$x = r\cos\theta$; area $\frac{1}{2}\int r^2 d\theta$

4 Induction

base case + inductive step

Check yourself

1. For $ax^2 + bx + c = 0$, what is $\alpha\beta$?
2. What is $\det\left(\begin{smallmatrix}3 & 1\\ 2 & 4\end{smallmatrix}\right)$?
3. Convert (r, θ) to the x-coordinate.
4. The two steps of proof by induction?



Answers **1.** c/a **2.** 10 **3.** $x = r\cos\theta$ **4.** base case, then inductive step