*Further Mathematics is marked overwhelmingly on method, so each solution here names the technique in its first line and then carries it out in full. Where a step is one examiners routinely find missing – an induction conclusion, a checked endpoint, a stated pivot – it is called out.* 本科目按方法给分，故每题先写明所用方法，再完整展开。

## Complex numbers and roots

### Find the three cube roots of 8i, giving each in the form a + bi.

$\sqrt{3} + i$, $-\sqrt{3} + i$, and $-2i$.

*Write 8i in modulus-argument form: modulus 8, argument π/2, so 8i = 8e^(iπ/2). By de Moivre the cube roots have modulus 8^(1/3) = 2 and arguments (π/2 + 2kπ)/3 for k = 0, 1, 2, that is π/6, 5π/6 and 3π/2. Converting: 2(cos π/6 + i sin π/6) = √3 + i; 2(cos 5π/6 + i sin 5π/6) = −√3 + i; and 2(cos 3π/2 + i sin 3π/2) = −2i. The roots are spaced 2π/3 apart on a circle of radius 2, which is the check to state.*

### The equation $x^3 - 5x^2 + 7x - 3 = 0$ has roots $\alpha$, $\beta$, $\gamma$. Find the equation whose roots are $\alpha + 2$, $\beta + 2$ and $\gamma + 2$.

$y^3 - 11y^2 + 39y - 45 = 0$.

*Do NOT find the roots. Substitute: if y = x + 2 then x = y − 2, so replace x in the original equation.* $(y-2)^3 - 5(y-2)^2 + 7(y-2) - 3 = 0$. *Expanding:* $(y^3 - 6y^2 + 12y - 8) - 5(y^2 - 4y + 4) + 7y - 14 - 3 = y^3 - 11y^2 + 39y - 45 = 0$. *The substitution method is what is being examined; solving the cubic and adding 2 to each root scores far less even when the answer is right.*

## Matrices

### Find the eigenvalues and corresponding eigenvectors of the matrix with rows (4, 1) and (2, 3).

$\lambda = 5$ with eigenvector (1, 1); $\lambda = 2$ with eigenvector (1, −2).

*Solve* $\det(A - \lambda I) = 0$: $(4 - \lambda)(3 - \lambda) - 2 = \lambda^2 - 7\lambda + 10 = 0$, *giving* $\lambda = 5$ *and* $\lambda = 2$. *For* $\lambda = 5$, $(A - 5I)x = 0$ *gives* $-x + y = 0$, *so y = x and an eigenvector is (1, 1). For* $\lambda = 2$, $2x + y = 0$, *so* $y = -2x$ *and an eigenvector is (1, −2). Any non-zero multiple is acceptable, and saying so avoids a needless normalisation. Check by multiplying: A(1,1) = (5,5) = 5(1,1).*

### Using the result above, find $A^5$ applied to the vector (3, 0), without computing $A^5$.

$(2 \cdot 5^5 + 2^5, 2 \cdot 5^5 - 2 \cdot 2^5) = (6282, 6186)$.

*Write the vector in the eigenbasis: (3, 0) = a(1, 1) + b(1, −2). From the components, a + b = 3 and a − 2b = 0, so b = 1 and a = 2. Applying A five times scales each eigenvector by its eigenvalue to the fifth power:* $A^5(3,0) = 2 \cdot 5^5(1,1) + 1 \cdot 2^5(1,-2) = 6250(1,1) + 32(1,-2) = (6282, 6186)$. *This is the point of diagonalisation – the matrix itself is never multiplied out.*

# Calculus and differential equations

### Given I_n as the integral of $x^n$e^(−x) from 0 to infinity, show that I_n = n · I_(n−1) and hence find $I_4$.

$I_4 = 24$.

*Integrate by parts with $u = x^n$ and $dv$ = e^(−x)dx: I_n = $[-x^n e$^(−x)$]_0$^∞ + n∫$x^{n-1}$e^(−x)dx. The boundary term vanishes at both ends – at infinity because the exponential dominates any power, and at zero because $x^n$ is zero for n ≥ 1 – so I_n = n · I_(n−1). The base case must be stated: $I_0$ = ∫e^(−x)dx = 1. Then $I_4 = 4 \cdot 3 \cdot 2 \cdot 1 \cdot I_0 = 24$. The base case is the step most often omitted and it is worth a mark on its own.*

### Solve $d^2y/dx^2$ − 4dy/dx + 4y = e^(2x), given y(0) = 0 and y'(0) = 1.

y = (x + ½$x^2$)e^(2x).

*The auxiliary equation $m^2 - 4m + 4 = 0$ has the repeated root m = 2, so the complementary function is (A + Bx)e^(2x). The right-hand side e^(2x) already appears in the complementary function, and so does xe^(2x), so the particular integral must be tried as $Cx^2$e^(2x) – this escalation is where the marks are. Differentiating and substituting gives 2C = 1, so C = ½. The general solution is y = (A + Bx + ½$x^2$)e^(2x). Applying y(0) = 0 gives A = 0; differentiating and applying y'(0) = 1 gives B = 1. Hence y = (x + ½$x^2$)e^(2x).*

# Proof and vectors

### Prove by induction that the sum of r · r! from r = 1 to n equals (n+1)! − 1 for all positive integers n.

Proved.

*Base case: for n = 1 the left side is 1 · 1! = 1 and the right side is 2! − 1 = 1, so the statement holds. Inductive step: assume the sum to k equals (k+1)! − 1. Then the sum to k+1 is (k+1)! − 1 + (k+1)(k+1)! = (k+1)![1 + (k+1)] − 1 = (k+1)!(k+2) − 1 = (k+2)! − 1, which is the statement with n = k+1. Conclusion: the statement is true for n = 1, and its truth for n = k implies its truth for n = k+1, so by induction it holds for all positive integers n. That final sentence is a mark in its own right and is the one most often left out.*

### Find the shortest distance between the lines r = (1,0,2) + s(2,1,−1) and r = (0,3,1) + t(1,−1,2).

√6 / 2, about 1.22.

*The lines are skew, so use the common perpendicular. Its direction is the vector product of the two direction vectors: (2,1,−1) × (1,−1,2) = (2 · 2 − (−1)(−1), (−1)(1) − 2 · 2, 2(−1) − 1 · 1) = (1, −5, −3). The vector joining a point on each line is (0,3,1) − (1,0,2) = (−1, 3, −1). The distance is the magnitude of the projection of that joining vector onto the common perpendicular: |(−1)(1) + 3(−5) + (−1)(−3)| / √(1 + 25 + 9) = |−1 − 15 + 3|/√35 = 13/√35 = 2.20. State the method first – it is the same three lines whatever the numbers, and naming it secures the method marks before any arithmetic.*