Further Pure Mathematics 2

A-Level Further Mathematics

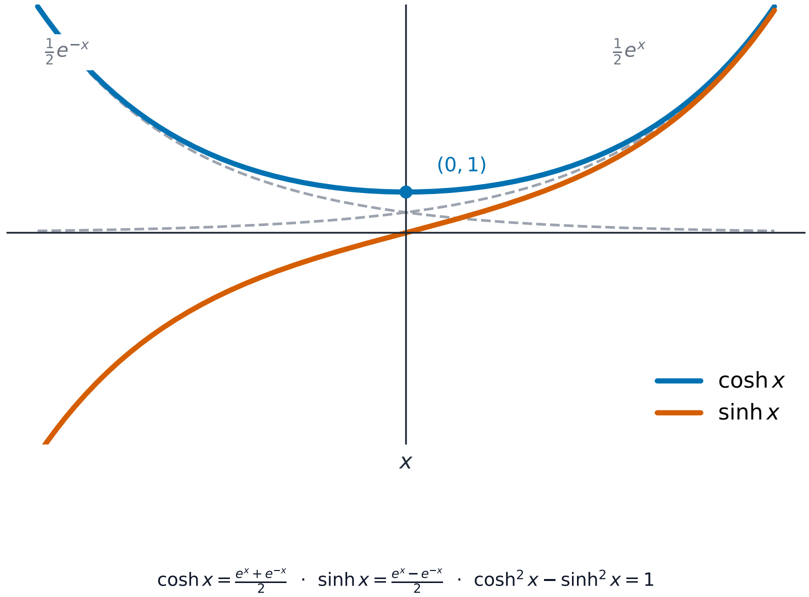
This handout covers Topic 2: **Further Pure Mathematics** 进阶纯数学 2. It adds hyperbolic functions, eigenvalues, new differentiation and integration, de Moivre's theorem, and methods for differential equations.

Hyperbolic functions

A hanging cable forms a catenary — the curve of the hyperbolic cosine.

The **hyperbolic functions** 双曲函数 are built from the exponential function:

$$\cosh x = \frac{e^x + e^{-x}}{2}, \qquad \sinh x = \frac{e^x - e^{-x}}{2}, \qquad \tanh x = \frac{\sinh x}{\cosh x}.$$



cosh *is the average of* e^x *and* e^{-x}*;* sinh *is half their difference.*

They obey identities much like the trigonometric ones, the main one being $\cosh^2 x - \sinh^2 x = 1$. The three **reciprocals** complete the set: $\operatorname{sech} x = \dfrac{1}{\cosh x}$, $\operatorname{cosech} x = \dfrac{1}{\sinh x}$, and $\coth x = \dfrac{1}{\tanh x} = \dfrac{\cosh x}{\sinh x}$ – all built from e^x too, and worth recognising in integrals and mark schemes. The **inverse hyperbolic functions** 反双曲函数 have a **logarithmic form** 对数形式, for example $\sinh^{-1} x = \ln\left(x + \sqrt{x^2+1}\right)$.

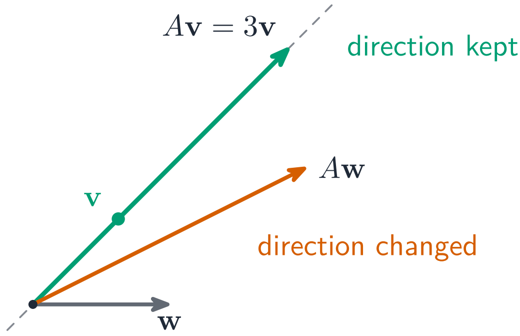
Worked example. Show that $\cosh^2 x - \sinh^2 x = 1$.

$$\cosh^2 x - \sinh^2 x = \frac{(e^x + e^{-x})^2 - (e^x - e^{-x})^2}{4} = \frac{(e^{2x} + 2 + e^{-2x}) - (e^{2x} - 2 + e^{-2x})}{4} = \frac{4}{4} = 1.$$

Matrices

You can write three linear equations in three unknowns as a single **matrix equation** 矩阵方程 $A\mathbf{x} = \mathbf{b}$. If A is non-singular there is one solution; if A is singular the equations are either inconsistent (no solution) or have infinitely many.

For a square matrix A, the **characteristic equation** 特征方程 is $\det(A - \lambda I) = 0$. Its solutions are the **eigenvalues** 特征值 λ, and for each one the vector $\mathbf{v}$ with $A\mathbf{v} = \lambda\mathbf{v}$ is an **eigenvector** 特征向量. You can then write $A = QDQ^{-1}$, where D is a **diagonal matrix** 对角矩阵 of eigenvalues and the columns of Q are the eigenvectors.



Multiplying an eigenvector by A only stretches it; a general vector is turned to a new direction.

Worked example. Find the eigenvalues of $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$.

The characteristic equation is

$$\det \begin{pmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{pmatrix} = (2-\lambda)^2 - 1 = 0 \;\Rightarrow\; 2-\lambda = \pm 1.$$

So $\lambda = 1$ or $\lambda = 3$. (The eigenvector for $\lambda = 3$ is $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and for $\lambda = 1$ is $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.)

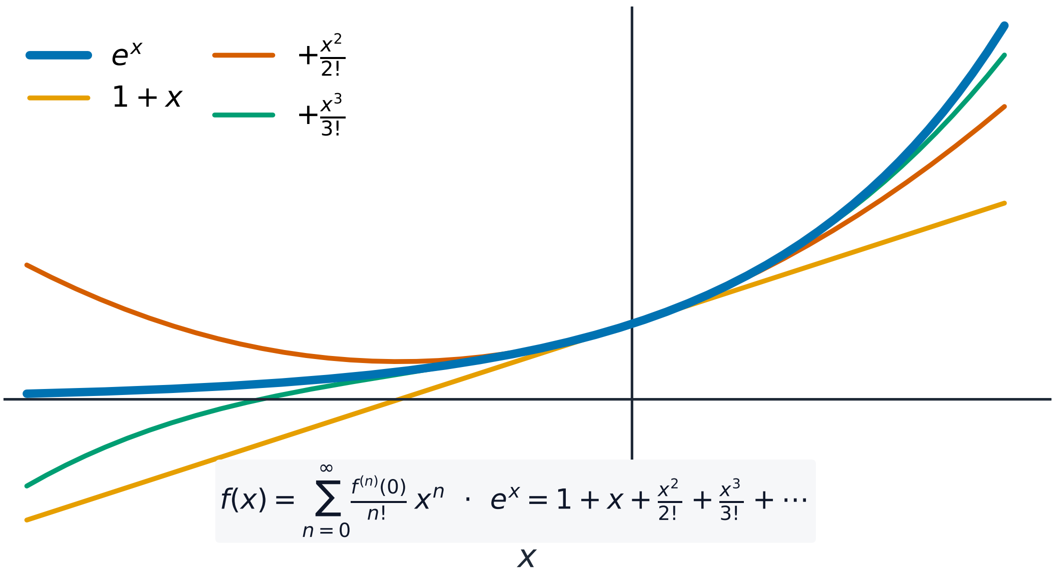
Differentiation

You can now differentiate the hyperbolic functions and the inverse functions $\sin^{-1} x$, $\tan^{-1} x$, $\sinh^{-1} x$ and so on. You can also find $\dfrac{d^2y}{dx^2}$ for curves given implicitly or parametrically.

A **Maclaurin's series** 麦克劳林级数 writes a function as a power series:

$$f(x) = f(0) + f'(0)\,x + \frac{f''(0)}{2!}\,x^2 + \frac{f'''(0)}{3!}\,x^3 + \cdots$$

For example $e^x = 1 + x + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + \cdots$



Adding more terms makes the polynomial match e^x over a wider stretch around $x = 0$.

Worked example. Find the Maclaurin series of $f(x) = \ln(1+x)$ up to the term in x^3.

$f(0) = 0$; $f'(x) = \dfrac{1}{1+x}$ so $f'(0) = 1$; $f''(x) = \dfrac{-1}{(1+x)^2}$ so $f''(0) = -1$; $f'''(x) = \dfrac{2}{(1+x)^3}$ so $f'''(0) = 2$. Substituting into the series:

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \cdots$$

Integration

Learn these standard integrals:

$$\int \frac{1}{a^2 + x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a} + C, \qquad \int \frac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\frac{x}{a} + C.$$

A **trigonometric substitution**, a **hyperbolic substitution**, or **completing the square** 配方法 in the denominator handles related forms. A **reduction formula** 递推公式 links an integral I_n to I_{n-1}, so you can work down step by step. Integration also gives the **arc length** 弧长 of a curve and the **surface area of revolution** 旋转曲面面积 when a curve is turned about an axis.

Worked example. Find $\displaystyle\int \frac{1}{\sqrt{4 - x^2}}\,dx$.

Here $a^2 = 4$, so $a = 2$ and

$$\int \frac{1}{\sqrt{4 - x^2}}\, dx = \sin^{-1}\frac{x}{2} + C.$$

Bounding a sum by an integral. Draw rectangles of unit width under, or over, a decreasing curve such as $y = \frac{1}{x}$. Each rectangle of height $\frac{1}{r}$ is trapped between two integral strips, so summing gives

$$\ln(n+1) < \sum_{r=1}^{n} \frac{1}{r} < 1 + \ln n.$$

Shrinking the rectangle width to $\frac{1}{n} \to 0$ turns such a sum into a **Riemann sum** 黎曼和 that equals the integral in the limit, e.g. $\lim_{n\to\infty} \frac{1}{n} \sum_{r=1}^{n} f\left(\frac{r}{n}\right) = \int_0^1 f(x)\, dx$.

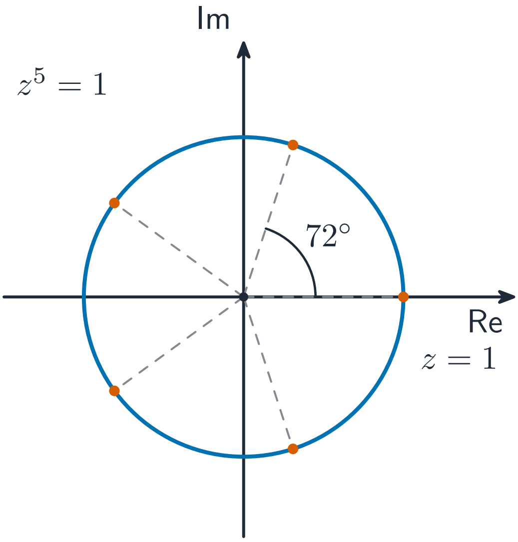
Complex numbers

Self-similar patterns like Romanesco broccoli arise from iterating functions in the complex plane.

A **complex number** 复数 in polar form is $z = r(\cos\theta + i\sin\theta)$. **De Moivre's theorem** 棣莫弗定理 says that for any integer n,

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta.$$

It is used to expand $\cos n\theta$ and $\sin n\theta$ in powers, to sum series, and to find the n **roots of unity** 单位根 (the solutions of $z^n = 1$, equally spaced around the unit circle).



The five solutions of $z^5 = 1$ are equally spaced 72° apart on the unit circle.

Worked example. Use de Moivre's theorem to express $\cos 3\theta$ in terms of $\cos\theta$.

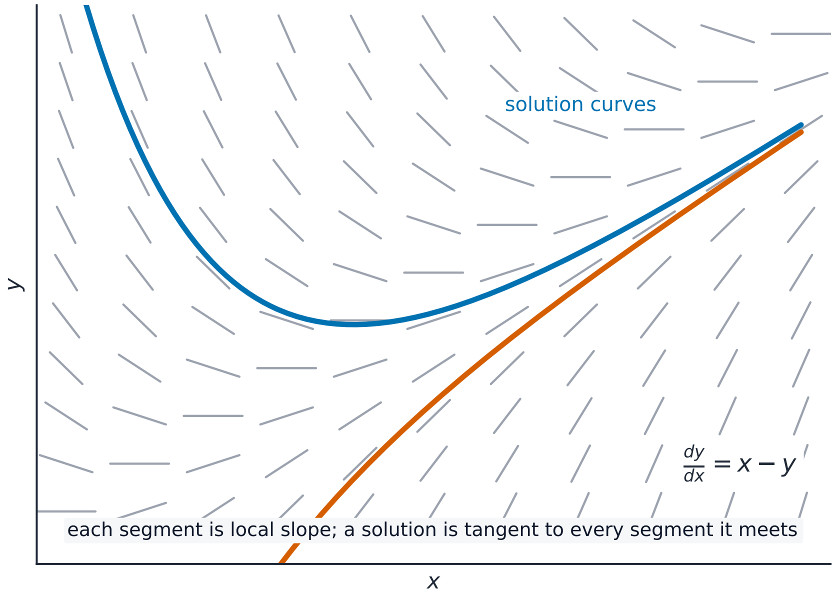
Take the real part of $(\cos\theta + i\sin\theta)^3 = \cos 3\theta + i\sin 3\theta$:

$$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta = \cos^3\theta - 3\cos\theta(1-\cos^2\theta) = 4\cos^3\theta - 3\cos\theta.$$

Differential equations

For a first order linear equation $\dfrac{dy}{dx} + P(x)\,y = Q(x)$, multiply by the **integrating factor** 积分因子 $\mu = e^{\int P\,dx}$. The left side then becomes $\dfrac{d}{dx}(\mu y)$, so you can integrate directly.

For a linear equation with constant coefficients, the **general solution** 通解 is the sum of two parts: the **complementary function** 余函数 (the solution of the equation with the right side set to 0, found from the auxiliary equation) and a **particular integral** 特积分 (any one solution of the full equation). **Initial conditions** then fix the constants to give the **particular solution** 特解; simpler equations with **separable variables** 可分离变量 are integrated directly.



A solution curve follows the slope field, staying tangent to the little segments everywhere.

Worked example. Solve $\dfrac{dy}{dx} + 2y = e^x$.

The integrating factor is $\mu = e^{\int 2\,dx} = e^{2x}$. Multiplying through gives $\dfrac{d}{dx}(y\,e^{2x}) = e^{3x}$, so

$$y\,e^{2x} = \tfrac{1}{3}e^{3x} + C \;\Rightarrow\; y = \tfrac{1}{3}e^{x} + Ce^{-2x}.$$

Second order, constant coefficients. For $\dfrac{d^2y}{dx^2} + b\dfrac{dy}{dx} + cy = f(x)$, solve the **auxiliary equation** 辅助方程 $m^2 + bm + c = 0$. The complementary function depends on its roots:

- two distinct real roots m_1, m_2: $y = Ae^{m_1x} + Be^{m_2x}$;
- a repeated root m: $y = (A + Bx)e^{mx}$;
- complex roots $p \pm qi$: $y = e^{px}(A\cos qx + B\sin qx)$.

Then add a particular integral by trying a form matching $f(x)$ (a polynomial, ae^{bx}, or $a\cos px + b\sin px$); if that trial already appears in the complementary function, **multiply it by** x.

Worked example. Solve $\dfrac{d^2y}{dx^2} - 4\dfrac{dy}{dx} + 4y = 0$. The auxiliary equation $m^2 - 4m + 4 = (m-2)^2 = 0$ has a repeated root $m = 2$, so $y = (A + Bx)e^{2x}$.

Exam tips

- Learn the **hyperbolic identities** and derivatives, and use the inverse hyperbolic functions in integration.
- Solve a linear **differential equation** as complementary function + particular integral, then apply the boundary conditions.
- Use **de Moivre's theorem** for powers and roots of complex numbers and to derive trigonometric identities.
- Reduce an integral to a standard form by a stated substitution.