Week 17

A-Level Further Mathematics

Homework bundle for this week

本周作业合集

exercises.lol

Contents 目录

2. Further Pure Mathematics 2

Starter Quiz · 课前小测

A-Level Further Mathematics   Name: ____________________   Score: ________

1. Using cosh x = $(e^x + e^{-x})/2$, what is cosh 0?

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2. The matrix [[2,1],[1,2]] has characteristic equation $(2-\lambda)^2 - 1 = 0$, giving $\lambda = 1$ or $\lambda = 3$. What is the larger eigenvalue?

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3. The derivative of sinh x is:

   ☐ cosh x

   ☐ −cosh x

   ☐ sinh x

   ☐ tanh x

4. What is $\int 1/(a^2 + x^2)\,dx$?

   ☐ $(1/a)\tan^{-1}(x/a) + C$

   ☐ $\sin^{-1}(x/a) + C$

   ☐ $\ln(a^2 + x^2) + C$

   ☐ $a\tan^{-1}(ax) + C$

5. De Moivre's theorem states $(\cos\theta + i\sin\theta)^n$ equals:

   ☐ $\cos n\theta + i\sin n\theta$

   ☐ $n(\cos\theta + i\sin\theta)$

   ☐ $\cos\theta^n + i\sin\theta^n$

   ☐ $\cos(\theta/n) + i\sin(\theta/n)$

6. For dy/dx + 2y = e^x, the integrating factor μ = e^($\int$2 dx) is:

   ☐ e^(2x)

   ☐ e^x

   ☐ 2x

   ☐ e^(x^2)

7. For any x, what does $\cosh^2 x - \sinh^2 x$ equal?

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8. The eigenvalues of a matrix A are found by ng:

- ☐ $\det(A - \lambda I) = 0$
- ☐ $A = 0$
- ☐ $\det(A) = 1$
- ☐ $Av = 0$

9. The derivative of $\tan^{-1}x$ is $1/(1+x^2)$. What is its value at $x = 0$?

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10. How many distinct roots of unity does the equation $z^5 = 1$ have?

    ____

1. **1**

   cosh 0 = $(e^0 + e^0)/2 = (1 + 1)/2 = 1$.

2. **3**

   $(2-\lambda)^2 = 1 \rightarrow 2-\lambda = \pm 1 \rightarrow \lambda = 1$ or 3; the larger is 3.

3. **cosh x**

   d/dx(sinh x) = cosh x (and d/dx(cosh x) = sinh x).

4. **$(1/a)\ \tan^{-1}(x/a) + C$**

   This is the standard inverse-tangent integral.

5. **cos nθ + i sin nθ**

   Raising to the power n multiplies the argument by n: cos nθ + i sin nθ.

6. **e^(2x)**

   $\int 2$ dx = 2x, so μ = e^(2x).

7. **1**

   The hyperbolic identity is $\cosh^2 x - \sinh^2 x = 1$.

8. **$\det(A - \lambda I) = 0$**

   The characteristic equation $\det(A - \lambda I) = 0$ gives the eigenvalues.

9. **1**

   $1/(1+0^2) = 1$.

10. **5**

    $z^n = 1$ has exactly n roots, equally spaced on the unit circle; here n = 5.

A-LEVEL FURTHER MATHEMATICS • EXERCISE SHEET

2.1 Hyperbolic functions

Name: ______________ Class: __________ Date: ____________ **Total: 13 marks**

KEY WORDS hyperbolic functions 双曲函数 • logarithmic form 对数形式

Objective

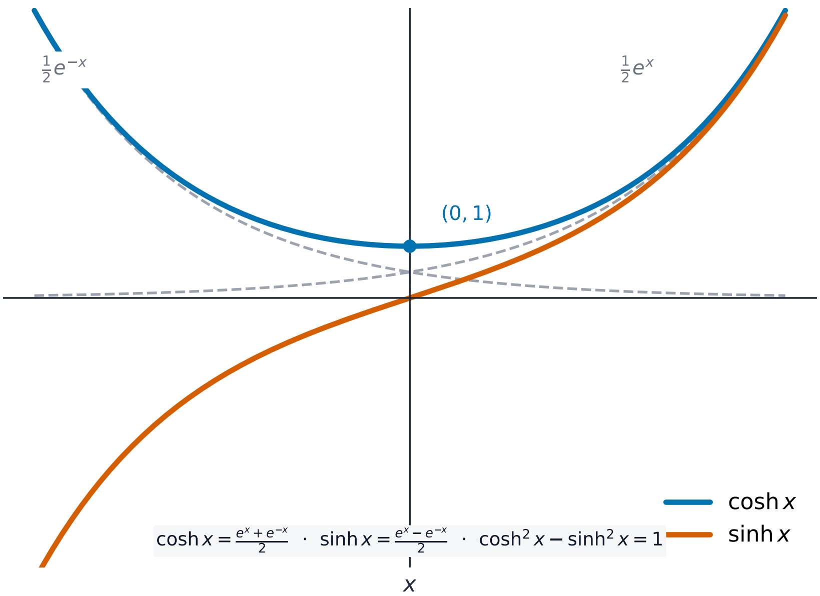
Build the skills to answer exam questions on **hyperbolic functions** 双曲函数 — the definitions, identities, and the **logarithmic form** 对数形式 of the inverses.

You must be able to:

- use $\cosh x = \frac{1}{2}(e^x + e^{-x})$, $\sinh x = \frac{1}{2}(e^x - e^{-x})$
- use $\cosh^2 x - \sinh^2 x = 1$ and related identities
- use $\sinh^{-1} x = \ln(x + \sqrt{x^2 + 1})$ and solve equations

1 Worked examples

■ Identity & solving



cosh *is the average of* e^x *and* e^{-x}; sinh *is half their difference*

$$\cosh^2 x - \sinh^2 x = \frac{(e^x + e^{-x})^2 - (e^x - e^{-x})^2}{4} = \frac{4}{4} = 1.$$

Solve $\cosh x = 2$**:** $\dfrac{e^x + e^{-x}}{2} = 2 \Rightarrow e^{2x} - 4e^x + 1 = 0 \Rightarrow e^x = 2 \pm \sqrt{3}$, $x = \ln(2 \pm \sqrt{3})$.

2 Practice

2.1 Write down the value of $\cosh 0$. [1]

2.2 Evaluate $\sinh^{-1} 0$ using the logarithmic form. [1]

3 Exam-style questions

3.1 The identity satisfied by hyperbolic functions is: [1]

- **A** $\cosh^2 x + \sinh^2 x = 1$
- **B** $\cosh^2 x - \sinh^2 x = 1$
- **C** $\sinh^2 x - \cosh^2 x = 1$
- **D** $\cosh^2 x - \sinh^2 x = -1$

3.2 Solve the equation $\cosh x = 3$, giving exact answers. [4]

3.3 (a) Show that $\sinh^{-1} x = \ln(x + \sqrt{x^2 + 1})$. [4]

(b) Hence evaluate $\sinh^{-1}(\frac{3}{4})$ as a single logarithm. [2]

Solutions

2.1 $\cosh 0 = \frac{1}{2}(1+1) = 1$.

2.2 $\sinh^{-1} 0 = \ln(0 + \sqrt{0+1}) = \ln 1 = 0$.

3.1 **B**. $\cosh^2 x - \sinh^2 x = 1$.

3.2 $\dfrac{e^x + e^{-x}}{2} = 3 \Rightarrow e^{2x} - 6e^x + 1 = 0$; $e^x = \dfrac{6 \pm \sqrt{32}}{2} = 3 \pm 2\sqrt{2}$; $x = \ln(3 \pm 2\sqrt{2})$.

3.3 (a) let $y = \sinh^{-1} x$, so $x = \sinh y = \frac{1}{2}(e^y - e^{-y})$; $e^{2y} - 2xe^y - 1 = 0 \Rightarrow e^y = x + \sqrt{x^2+1}$ (positive root); $y = \ln(x + \sqrt{x^2+1})$.
(b) $\sinh^{-1} \frac{3}{4} = \ln\left(\frac{3}{4} + \sqrt{\frac{9}{16} + 1}\right) = \ln\left(\frac{3}{4} + \frac{5}{4}\right) = \ln 2$.

7 **(a)** Show that $\frac{\mathrm{d}}{\mathrm{d}x}\left(\tanh^{-1}x\right)=\frac{1}{1-x^2}$. [3]

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(b) Find the solution of the differential equation

$$x\frac{\mathrm{d}y}{\mathrm{d}x}-y=x^2\tanh^{-1}x,$$

for $0<x<1$, given that $y=0$ when $x=\frac{1}{2}$. Give your answer in an exact form. [9]

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2 **(a)** Find the exact values of x for which $\cosh 2x = 6\sinh^2 x$, giving your answers in logarithmic form. [4]

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(b) Sketch the curves $C_1 : y = \cosh 2x$ and $C_2 : y = 6\sinh^2 x$ on the same diagram. [2]

6 **(a)** Starting from the definitions of tanh and sech in terms of exponentials, prove that

$$1-\tanh^2 u = \operatorname{sech}^2 u. \qquad [3]$$

(b) Show that $\dfrac{\mathrm{d}}{\mathrm{d}t}(\operatorname{sech}^{-1} t) = -\dfrac{1}{t\sqrt{1-t^2}}$. [4]

It is given that

$$x = \tanh^{-1} t \qquad \text{and} \qquad y = t\,\text{sech}^{-1} t, \qquad \text{for } 0 < t < 1.$$

(c) Show that $\dfrac{dy}{dx} = -\sqrt{1-t^2} + \left(1-t^2\right)\text{sech}^{-1} t$. [4]

(d) Find $\dfrac{d^2y}{dx^2}$ in terms of t. [4]

2 **(a)** Starting from the definitions of tanh and sech in terms of exponentials, prove that

$$\tanh^2 t + \operatorname{sech}^2 t = 1.$$ [3]

(b) The curve C has parametric equations

$$x = \ln(\cosh t), \quad y = \tan^{-1}(\sinh t), \quad \text{for } 0 \leqslant t \leqslant 1.$$

Find the length of C. [5]

3 The curve C has equation

$$9y^2 - 3\sinh^{-1}(xy) = 1 - 3\ln 3.$$

(a) Show that, at the point $(4, \frac{1}{3})$ on C, $\dfrac{dy}{dx} = -\dfrac{1}{2}$. [4]

(b) Find the value of $\frac{d^2y}{dx^2}$ at the point $(4, \frac{1}{3})$. [5]

2.2 Matrices (eigenvalues and eigenvectors)

Name: ______________ Class: __________ Date: ____________ **Total: 14 marks**

KEY WORDS eigenvalues 特征值 • characteristic equation 特征方程 • eigenvector 特征向量 • matrices 矩阵

Objective

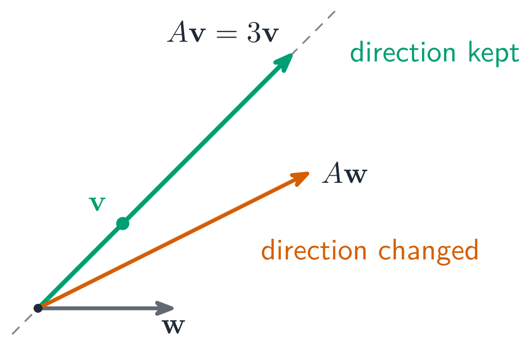
Build the skills to answer exam questions on **matrices** 矩阵 — solving $A\mathbf{x} = \mathbf{b}$, the **characteristic equation** 特征方程, **eigenvalues** 特征值 and **eigenvectors** 特征向量, and diagonalisation.

You must be able to:

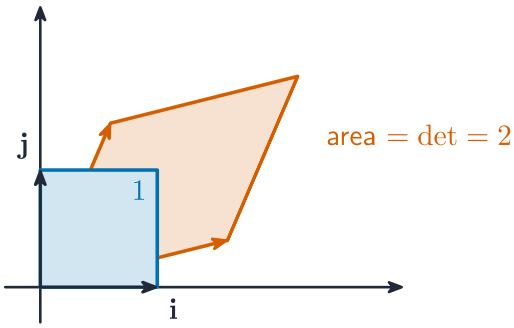
- find eigenvalues from $\det(A - \lambda I) = 0$
- find an eigenvector for each eigenvalue
- write $A = QDQ^{-1}$ and use it (e.g. for A^n)

1 Worked examples

■ Eigenvalues & eigenvectors



Multiplying an eigenvector by A only stretches it (by λ); a general vector changes direction



Eigenvectors are the directions a matrix leaves unturned

$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$: $\det(A - \lambda I) = (2-\lambda)^2 - 1 = 0 \Rightarrow \lambda = 1$ or 3. Eigenvector for $\lambda = 3$: $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$; for $\lambda = 1$: $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

2 Practice

2.1 Write down the characteristic equation of a matrix A. [1]

2.2 Find the eigenvalues of $\begin{pmatrix} 4 & 0 \\ 0 & -2 \end{pmatrix}$. [1]

3 Exam-style questions

3.1 A vector $\mathbf{v}$ is an eigenvector of A with eigenvalue λ when: [1]

- **A** $A\mathbf{v} = \mathbf{0}$
- **B** $A\mathbf{v} = \lambda\mathbf{v}$
- **C** $\det A = \lambda$
- **D** $A = \lambda I$

3.2 The matrix $A = \begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix}$.

(a) Find the eigenvalues. [2]

(b) Find an eigenvector for each eigenvalue. [4]

3.3 The matrix $M = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ has eigenvalues 1 and 3. Using $M = QDQ^{-1}$, find M^2 and verify by direct multiplication. [5]

Solutions

2.1 $\det(A - \lambda I) = 0$.

2.2 $\lambda = 4$ and $\lambda = -2$ (the diagonal entries).

3.1 **B**. $A\mathbf{v} = \lambda\mathbf{v}$ defines an eigenvector.

3.2 (a) $\det \begin{pmatrix} 3-\lambda & 1 \\ 0 & 2-\lambda \end{pmatrix} = (3-\lambda)(2-\lambda) = 0 \Rightarrow \lambda = 3, 2$.

(b) $\lambda = 3$: $(A - 3I)\mathbf{v} = 0 \Rightarrow \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}\mathbf{v} = 0 \Rightarrow \mathbf{v} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$; $\lambda = 2$: $\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\mathbf{v} = 0 \Rightarrow \mathbf{v} = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

3.3 $D = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$, so $M^2 = QD^2Q^{-1}$ with $D^2 = \begin{pmatrix} 1 & 0 \\ 0 & 9 \end{pmatrix}$; this gives $M^2 = \begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix}$; direct: $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}^2 = \begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix}$ ✓.

1 Find the values of k for which the system of equations

$$\begin{aligned} x - y + z &= 0, \\ x + ky + 3z &= 0, \\ x + 2y + kz &= 0, \end{aligned}$$

does not have a unique solution. [3]

6 The matrix $\mathbf{P}$ has non-zero eigenvalues and is given by

$$\mathbf{P} = \begin{pmatrix} a & 1 & 1 \\ 0 & 2a & -1 \\ 0 & 0 & -3a \end{pmatrix}.$$

(a) State, in terms of a, the eigenvalues of $\mathbf{P}$. [1]

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(b) **(i)** Find $\mathbf{P}^2$ in terms of a. [1]

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(ii) Use the characteristic equation of $\mathbf{P}$ to find $\mathbf{P}^{-1}$ in terms of a. [3]

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The 3×3 matrix **A** has eigenvalues 1, 2, 3 with corresponding eigenvectors

$$\begin{pmatrix} a \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 2a \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ -1 \\ -3a \end{pmatrix},$$

respectively.

(c) Find **A** in terms of a. [5]

8 **(a)** Find the values of k for which the system of equations

$$\begin{aligned} x - y + 2kz &= 1, \\ kx + y + 2z &= 2, \\ 2x - y + z &= 3, \end{aligned}$$

does not have a unique solution. [3]

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(b) Given that $k = -\frac{1}{2}$, show that the system of equations in part **(a)** is inconsistent. Interpret this situation geometrically. [3]

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(c) Given instead that $k = -1$, show that the system of equations in part **(a)** is also inconsistent. Interpret this situation geometrically. [4]

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The matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{pmatrix} 1 & -1 & -2 \\ -1 & 1 & 2 \\ 2 & -1 & 1 \end{pmatrix}.$$

(d) Use the characteristic equation of $\mathbf{A}$ to show that $\mathbf{A}^4 = p\mathbf{A}^2 + q\mathbf{A}$ where p and q are integers to be determined. [5]

Additional page

If you use the following page to complete the answer(s) to any question(s), the question number(s) must be clearly shown.

8 **(a)** Find the set of values of a for which the system of equations

$$\begin{aligned} 5x + ay \quad &= 3, \\ 20x - 5y \quad &= 2, \\ 2x - 3y + z &= 1, \end{aligned}$$

has a unique solution and interpret this situation geometrically. [3]

(b) Given that $a = -\frac{5}{4}$, show that the system of equations in part **(a)** is inconsistent and interpret this situation geometrically. [3]

The matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{pmatrix} 5 & 0 & 0 \\ 20 & -5 & 0 \\ 2 & -3 & 1 \end{pmatrix}.$$

(c) Find a matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that $(\mathbf{A}+3\mathbf{I})^2 = \mathbf{PDP}^{-1}$. [7]

Additional page

If you use the following page to complete the answer to any question, the question number must be clearly shown.

8 **(a)** It is given that λ is an eigenvalue of the non-singular square matrix $\mathbf{A}$, with corresponding eigenvector $\mathbf{e}$.

Show that $\mathbf{e}$ is an eigenvector of $\mathbf{A}^3$ with corresponding eigenvalue λ^3. [2]

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The matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{pmatrix} -1 & 3 & 4 \\ 0 & 1 & 0 \\ 0 & -2 & 5 \end{pmatrix}.$$

(b) Show that the eigenvalues of $\mathbf{A}$ are -1, 1 and 5. [2]

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(c) Find a matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{A}-2\mathbf{I}=\mathbf{PDP}^{-1}$. [6]

(d) Use the characteristic equation of $\mathbf{A}$ to show that $(\mathbf{A}-2\mathbf{I})^3 = a\mathbf{A}^2 + b\mathbf{A} + c\mathbf{I}$ where a, b and c are constants to be determined. [3]

Additional page

If you use the following page to complete the answer to any question, the question number must be clearly shown.

8 **(a)** Find the values of a for which the system of equations

$$\begin{aligned} \tfrac{3}{2}x + 3y + 8z &= 1, \\ ax + 3y + 4z &= 2, \\ ay - z &= 3, \end{aligned}$$

does not have a unique solution. [3]

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The matrix $\mathbf{A}$ is given by

$$\mathbf{A} = \begin{pmatrix} \frac{3}{2} & 3 & 8 \\ 0 & 3 & 4 \\ 0 & 0 & -1 \end{pmatrix}.$$

(b) Given that $\mathbf{B} = \mathbf{A}^{-1}$, use the characteristic equation of $\mathbf{A}$ to show that $\mathbf{B}^2 = p\mathbf{I} + q\mathbf{A}$, where p and q are constants to be determined. [4]

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(c) Find a matrix $\mathbf{P}$ and a diagonal matrix $\mathbf{D}$ such that $\mathbf{A}^{-1} = \mathbf{PDP}^{-1}$. [7]

Additional page

If you use the following page to complete the answer to any question, the question number must be clearly shown.

2.3 Differentiation (Maclaurin, hyperbolic, parametric)

Name: ____________ Class: ________ Date: __________ **Total: 14 marks**

KEY WORDS maclaurin's series 麦克劳林级数 • hyperbolic functions 双曲函数 • further differentiation 微分

Objective

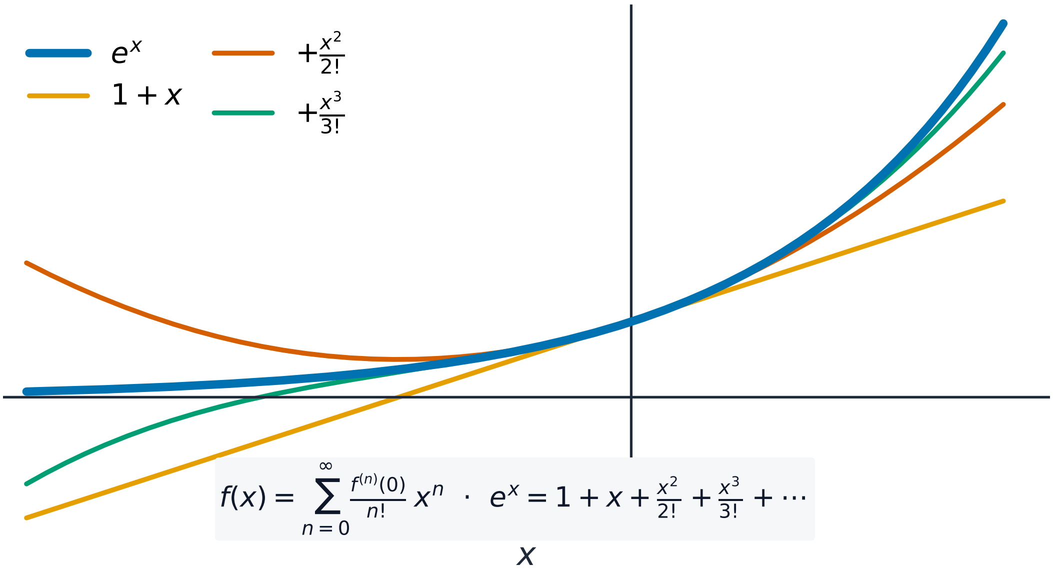
Build the skills to answer exam questions on **further differentiation 微分** — derivatives of inverse and **hyperbolic functions 双曲函数**, second derivatives of implicit/parametric curves, and **Maclaurin's series 麦克劳林级数**.

You must be able to:

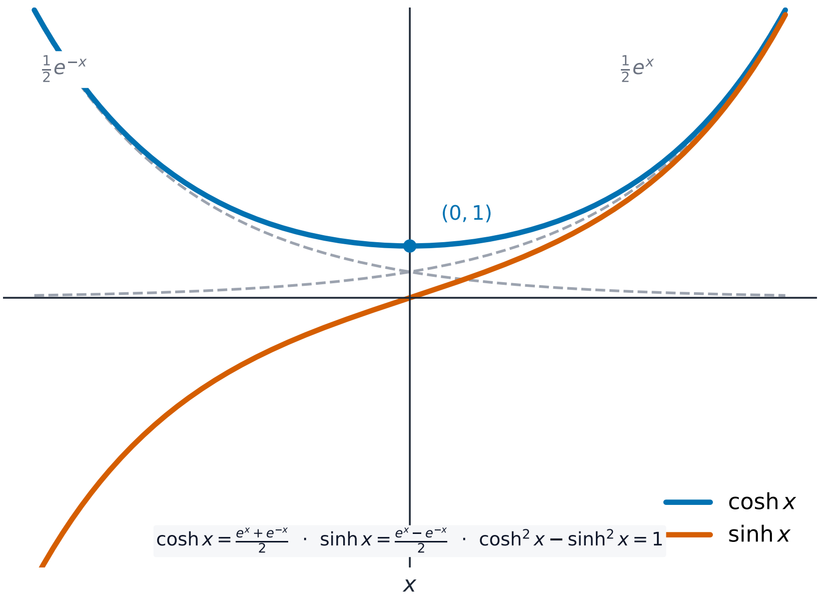
- differentiate $\sin^{-1} x$, $\tan^{-1} x$, $\sinh x$, $\cosh x$, $\tanh x$
- find $\dfrac{d^2y}{dx^2}$ for implicit/parametric curves
- find a Maclaurin series for a function

1 Worked examples

■ Maclaurin series



Adding more terms makes the polynomial match the function over a wider stretch around $x = 0$



The hyperbolic functions whose derivatives appear here

Maclaurin: $f(x) = f(0) + f'(0)x + \dfrac{f''(0)}{2!}x^2 + \cdots$; e.g. $e^x = 1 + x + \dfrac{x^2}{2!} + \cdots$.

Derivatives: $\dfrac{d}{dx}\sinh x = \cosh x$; $\dfrac{d}{dx}\tan^{-1} x = \dfrac{1}{1 + x^2}$.

2 Practice

2.1 Differentiate $y = \cosh(3x)$. [2]

2.2 Differentiate $y = \sin^{-1}(2x)$. [2]

3 Exam-style questions

3.1 $\dfrac{d}{dx}\tanh x$ equals: [1]

- **A** $\operatorname{sech}^2 x$
- **B** $\operatorname{cosech}^2 x$
- **C** $\cosh^2 x$
- **D** $\sinh^2 x$

3.2 Find the Maclaurin series for $\ln(1+x)$ up to and including the term in x^3. [4]

3.3 A curve is given by $x = t^2$, $y = t^3$. Find $\dfrac{d^2y}{dx^2}$ in terms of t. [5]

Solutions

2.1 $\dfrac{dy}{dx} = 3\sinh(3x)$.

2.2 $\dfrac{dy}{dx} = \dfrac{2}{\sqrt{1-4x^2}}$.

3.1 **A**. $\dfrac{d}{dx}\tanh x = \operatorname{sech}^2 x$.

3.2 $f(0) = 0$; $f'(x) = \dfrac{1}{1+x}$, $f'(0) = 1$; $f''(x) = -\dfrac{1}{(1+x)^2}$, $f''(0) = -1$; $f'''(0) = 2$; $\ln(1+x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \cdots$.

3.3 $\dfrac{dx}{dt} = 2t$, $\dfrac{dy}{dt} = 3t^2$, so $\dfrac{dy}{dx} = \dfrac{3t^2}{2t} = \dfrac{3t}{2}$; $\dfrac{d^2y}{dx^2} = \dfrac{d}{dt}\left(\dfrac{3t}{2}\right) \div \dfrac{dx}{dt} = \dfrac{3/2}{2t} = \dfrac{3}{4t}$.

2 The curve C has parametric equations

$$x = \sinh t \quad \text{and} \quad y = t + \cosh t.$$

(a) Find $\dfrac{dy}{dx}$ in terms of t. [2]

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(b) Show that $\dfrac{d^2y}{dx^2} = \dfrac{1 - \sinh t}{\cosh^3 t}$. [4]

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(c) Find the Maclaurin's series for y in terms of x up to and including the term in x^2. [2]

3 The variables x and y are such that $y = 2$ when $x = -1$ and

$$x^2y + (x+y)^3 = 3.$$

(a) Show that $\frac{dy}{dx} = \frac{1}{4}$ when $x = -1$. [3]

(b) Find the value of $\frac{d^2y}{dx^2}$ when $x = -1$. [6]

7 **(a)** Show that $\frac{d}{dx}\left(\frac{1}{2}x\sqrt{4-x^2}+2\sin^{-1}\left(\frac{1}{2}x\right)\right)=\sqrt{4-x^2}$. [3]

(b) Find the solution of the differential equation

$$2\frac{dy}{dx}+\frac{y}{2+x}=2\sqrt{2-x}$$

for which $y=\frac{1}{2}$ when $x=1$. Give your answer in an exact form. [8]

7 The curve C has parametric equations

$$x = 8\ln(\tan\tfrac{1}{2}t) - 3\cot t - 3t, \qquad y = 6\ln(\tan\tfrac{1}{2}t) + 4\cot t + 4t, \qquad \text{for } \tfrac{1}{6}\pi \leqslant t \leqslant \tfrac{1}{3}\pi.$$

(a) (i) Show that $\dfrac{dx}{dt} = 8\operatorname{cosec} t + 3\cot^2 t$. [1]

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(ii) Find $\dfrac{dy}{dt}$. [1]

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(iii) Find the exact value of the length of C. [6]

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(b) Show that

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = \frac{a\cot t\,\mathrm{cosec}\,t(\mathrm{cosec}^2 t + 1)}{(b\,\mathrm{cosec}\,t + c\cot^2 t)^n},$$

where a, b, c and n are integers to be determined. [5]

1 Find the Maclaurin's series for $\mathrm{e}^{\left(\frac{1}{x+2}\right)}$ up to and including the term in x^2. [5]

A-LEVEL FURTHER MATHEMATICS • EXERCISE SHEET

2.4 Integration (standard forms, reduction formulae)

Name: ______________ Class: __________ Date: ____________ **Total: 16 marks**

KEY WORDS completing the square 配方法 • reduction formula 递推公式 • further integration 积分 • arc length 弧长

Objective

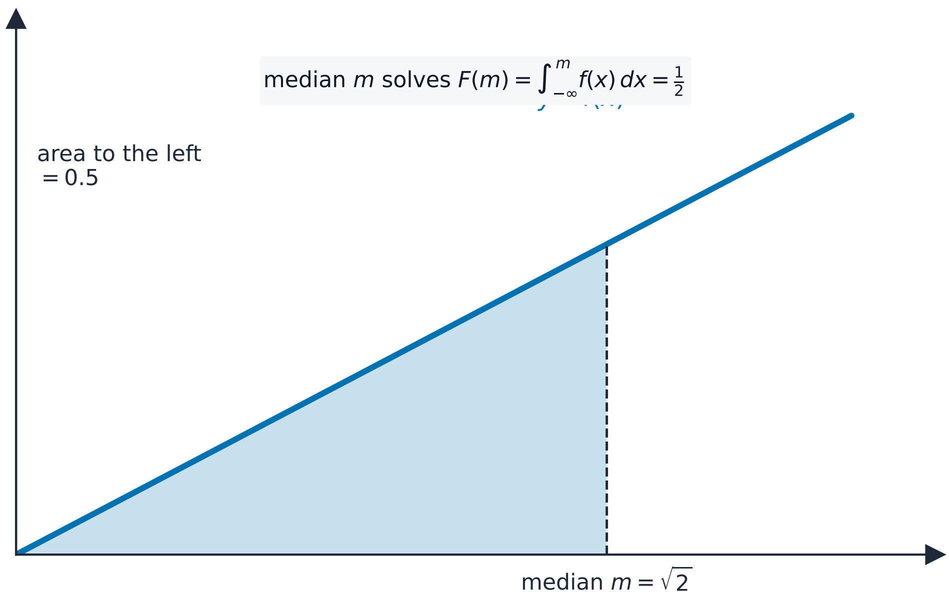
Build the skills to answer exam questions on **further integration 积分** — the standard inverse-function integrals, **reduction formulae 递推公式**, and **arc length 弧长**.

You must be able to:

- use $\int \frac{1}{a^2+x^2}\,dx$ and $\int \frac{1}{\sqrt{a^2-x^2}}\,dx$ (and completing the square)
- derive and use a reduction formula
- find an arc length

1 Worked examples

■ **Standard integrals & reduction**



An integral is the area under a curve; the inverse-function forms handle $\frac{1}{a^2+x^2}$ *and* $\frac{1}{\sqrt{a^2-x^2}}$

$$\int \frac{1}{\sqrt{4-x^2}}\,dx = \sin^{-1}\frac{x}{2} + C.$$

Reduction: $I_n = \int_0^{\pi/2} \sin^n x\,dx$ satisfies $I_n = \dfrac{n-1}{n} I_{n-2}$.

2 Practice

2.1 Find $\displaystyle\int \frac{1}{9+x^2}\,dx.$ [2]

2.2 Find $\displaystyle\int \frac{1}{\sqrt{25-x^2}}\,dx.$ [2]

3 Exam-style questions

3.1 $\int_0^2 \frac{1}{4+x^2}\,dx$ equals: [1]

- **A** $\frac{\pi}{8}$
- **B** $\frac{\pi}{4}$
- **C** $\frac{\pi}{2}$
- **D** π

3.2 Let $I_n = \int_0^{\pi/2} \sin^n x\,dx$.

(a) Using integration by parts, show that $I_n = \frac{n-1}{n} I_{n-2}$ for $n \geqslant 2$. [4]

(b) Hence evaluate I_4. [3]

3.3 Find $\int \frac{1}{x^2 - 4x + 13}\,dx$ by completing the square. [4]

Solutions

2.1 $\frac{1}{3}\tan^{-1}\dfrac{x}{3}+C.$

2.2 $\sin^{-1}\dfrac{x}{5}+C.$

3.1 **A**. $\left[\frac{1}{2}\tan^{-1}\frac{x}{2}\right]_0^2=\frac{1}{2}\tan^{-1}1=\frac{1}{2}\cdot\frac{\pi}{4}=\dfrac{\pi}{8}.$

3.2 (a) $I_n=\int_0^{\pi/2}\sin^{n-1}x\sin x\,dx$; by parts ($u=\sin^{n-1}x$, $dv=\sin x\,dx$): $I_n=[-\sin^{n-1}x\cos x]_0^{\pi/2}+(n-1)\int_0^{\pi/2}\sin^{n-2}x\cos^2 x\,dx$; the boundary term is 0; $\cos^2 x=1-\sin^2 x$ gives $I_n=(n-1)(I_{n-2}-I_n)$; so $nI_n=(n-1)I_{n-2}$, i.e. $I_n=\dfrac{n-1}{n}I_{n-2}.$

(b) $I_0=\dfrac{\pi}{2}$; $I_2=\frac{1}{2}I_0=\dfrac{\pi}{4}$; $I_4=\frac{3}{4}I_2=\dfrac{3\pi}{16}.$

3.3 $x^2-4x+13=(x-2)^2+9$; $\displaystyle\int\frac{1}{(x-2)^2+9}\,dx=\frac{1}{3}\tan^{-1}\frac{x-2}{3}+C.$

3 The integral I_n is defined by $I_n = \int_0 \left(1+x^2\right)\,\mathrm{d}x$.

(a) By considering $\frac{\mathrm{d}}{\mathrm{d}x}\left(x(1+x^2)^n\right)$, or otherwise, show that

$$(2n+1)I_n = 2^n + 2nI_{n-1}.$$

[5]

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(b) Find the exact value of I_{-2}. [4]

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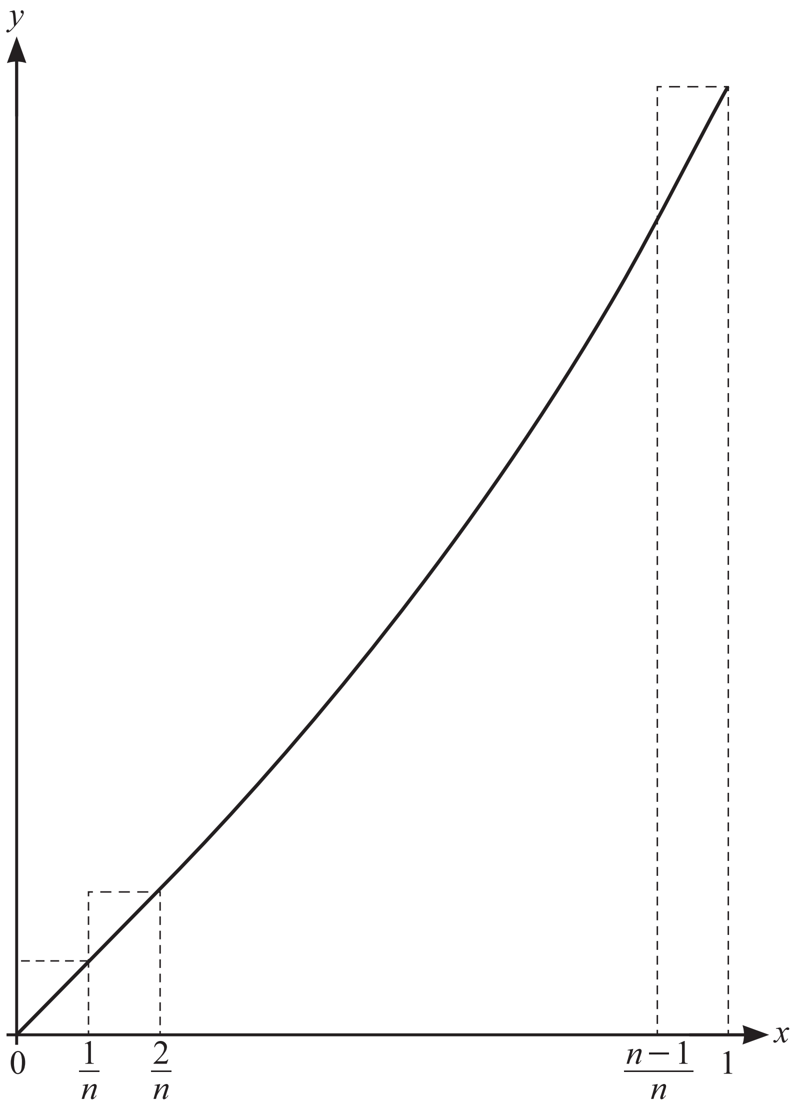

The diagram shows the curve with equation $y = \frac{1}{3}x^3 + x$ for $0 \leqslant x \leqslant 1$, together with a set of n rectangles of width $\frac{1}{n}$.

(a) By considering the sum of the areas of these rectangles, show that $\int_0^1 \left(\frac{1}{3}x^3 + x\right) dx < U_n$, where

$$U_n = \frac{1}{12}\left(1 + \frac{1}{n}\right)\left(7 + \frac{1}{n}\right).$$ [5]

(b) Use a similar method to find, in terms of n, a lower bound L_n for $\int_0^1 \left(\frac{1}{3}x^3 + x\right) \mathrm{d}x$. [4]

(c) Show that $\lim_{n \to \infty}(U_n - L_n) = 0$. [2]

1 The curve C has polar equation $r = e^{\frac{4}{3}\theta}$ for $0 \leqslant \theta \leqslant \pi$.

Find the length of C. Give your answer in an exact form. [4]

6 **(a)** Use the substitution $x = \frac{1}{2}\sqrt{2}\sinh u$ to find $\int \frac{1}{\sqrt{2x^2+1}}\,dx$. [3]

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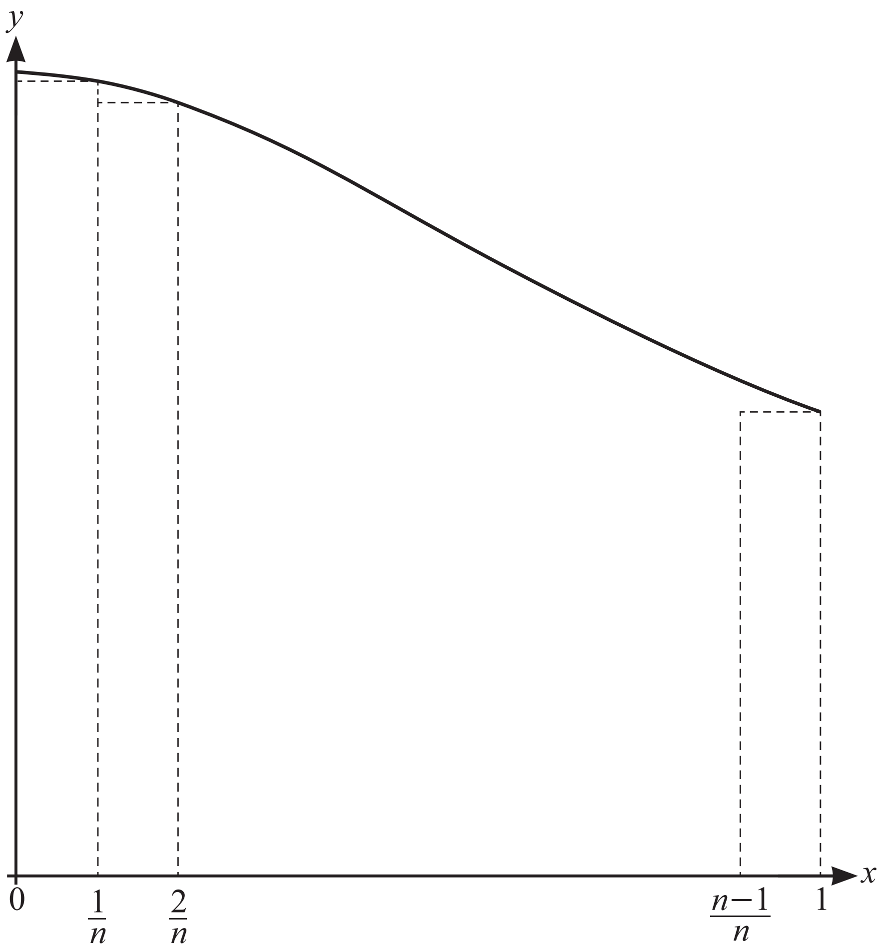

The diagram shows the curve with equation $y = \frac{1}{\sqrt{2x^2+1}}$ for $0 \leqslant x \leqslant 1$, together with a set of n rectangles of width $\frac{1}{n}$.

(b) By considering the sum of the areas of these rectangles, show that

$$\sum_{r=1}^{n} \frac{1}{\sqrt{2r^2+n^2}} < \frac{1}{2}\sqrt{2}\ln\left(\sqrt{2}+\sqrt{3}\right).$$ [5]

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(c) Use a similar method to find, in terms of n, a lower bound for $\sum_{r=1}^{n} \frac{1}{\sqrt{2r^2 + n^2}}$. [4]

(d) Deduce the exact value of $\lim_{n \to \infty} \sum_{r=1}^{n} \frac{1}{\sqrt{2r^2 + n^2}}$. [1]

1 It is given that $I_n = \int_0 x^n \sinh x \, dx$, where n is a non-negative integer.

(a) Show that, for $n \geqslant 2$,

$$I_n = \cosh 1 - n \sinh 1 + n(n-1) I_{n-2}.$$ [3]

(b) Find the exact value of I_2. [2]

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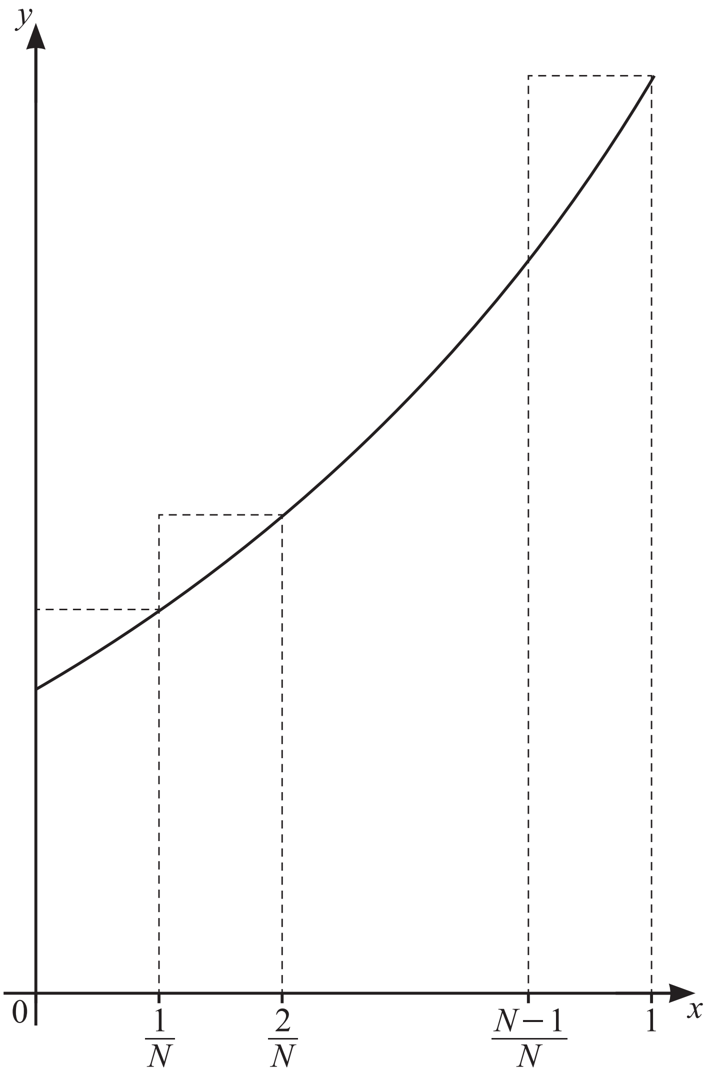

The diagram shows the curve with equation $y = \frac{1}{2}(3^x)$ for $0 \leqslant x \leqslant 1$, together with a set of N rectangles each of width $\frac{1}{N}$.

(a) By considering the sum of the areas of these rectangles, show that $\int_0^1 \frac{1}{2}(3^x)\,dx < U_N$, where

$$U_N = \frac{3^{\frac{1}{N}}}{N\left(3^{\frac{1}{N}} - 1\right)}.$$ [4]

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(b) Use a similar method to find, in terms of N, a lower bound L_{N} for $\int_0^1 \frac{1}{2}\left(3^x\right)\mathrm{d}x$. [4]

(c) By simplifying $U_N - L_N$, show that $\lim\limits_{N \to \infty}\left(U_N - L_N\right) = 0$. [2]

2 Let $I_n = \int_0^1 (1-x)^n \sinh x \, dx$, where n is a non-negative integer.

(a) Show that, for $n \geqslant 2$, $I_n = -1 + n(n-1)I_{n-2}$. [4]

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(b) Find the exact value of I_2. [3]

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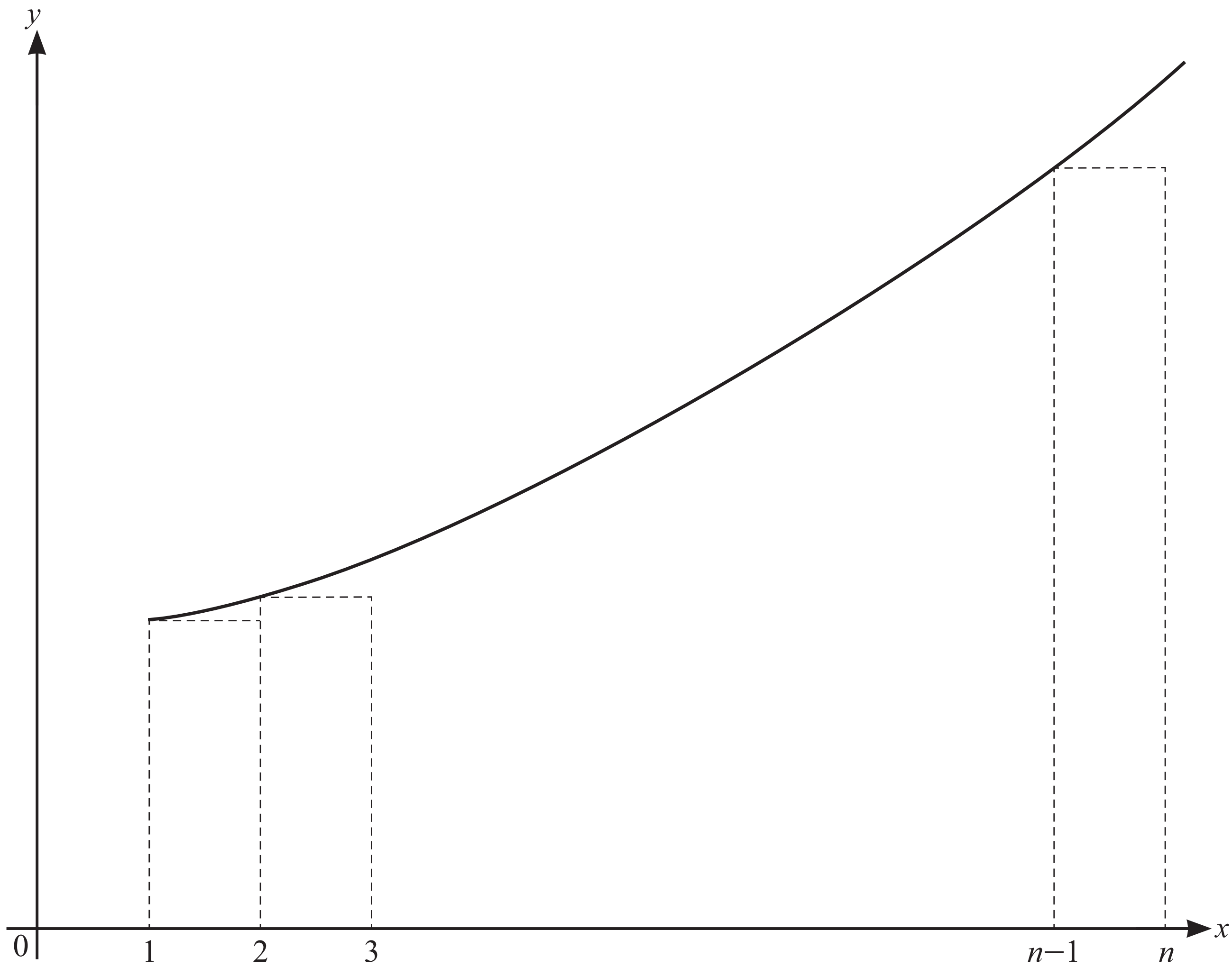

The diagram shows the curve with equation $y = \frac{1}{\sqrt{x}}e^{\sqrt{x}}$ for $x \geqslant 1$, together with a set of $n-1$ rectangles of unit width.

(a) By considering the sum of the areas of these rectangles, show that

$$\sum_{r=1}^{n} \frac{1}{\sqrt{r}}e^{\sqrt{r}} < \left(2 + \frac{1}{\sqrt{n}}\right)e^{\sqrt{n}} - 2e.$$ [5]

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(b) Use a similar method to find, in terms of n, a lower bound for $\sum_{r=1}^{n} \frac{1}{\sqrt{r}} e^{\sqrt{r}}$. [4]

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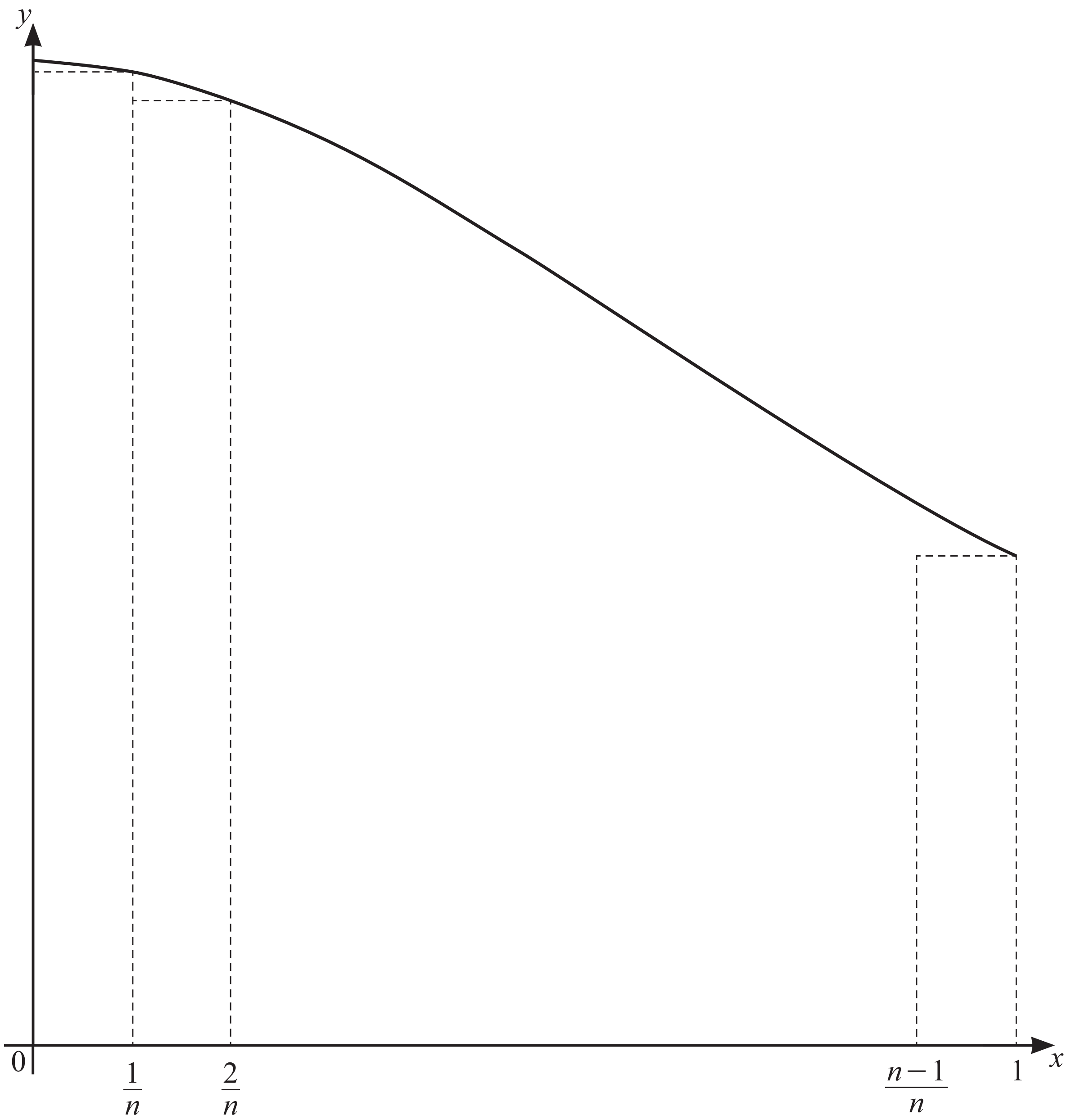


The diagram shows the curve with equation $y = \dfrac{1}{x^2+1}$ for $0 \leqslant x \leqslant 1$, together with a set of n rectangles of width $\dfrac{1}{n}$.

(a) By considering the sum of the areas of these rectangles, show that

$$\sum_{r=1}^{n} \frac{n}{n^2+r^2} < \frac{1}{4}\pi. \qquad [5]$$

(b) Use a similar method to find a lower bound for $\sum_{r=1}^{n} \frac{n}{n^2+r^2}$. Give your answer in terms of n and π. [4]

(c) Deduce the exact value of $\lim_{n\to\infty} \sum_{r=1}^{n} \frac{n}{n^2+r^2}$. [1]

A-LEVEL FURTHER MATHEMATICS • EXERCISE SHEET

2.5 Complex numbers (de Moivre's theorem)

Name: ____________ Class: ________ Date: __________ **Total: 15 marks**

KEY WORDS de moivre's theorem 棣莫弗定理 • roots of unity 单位根 • complex number 复数

Objective

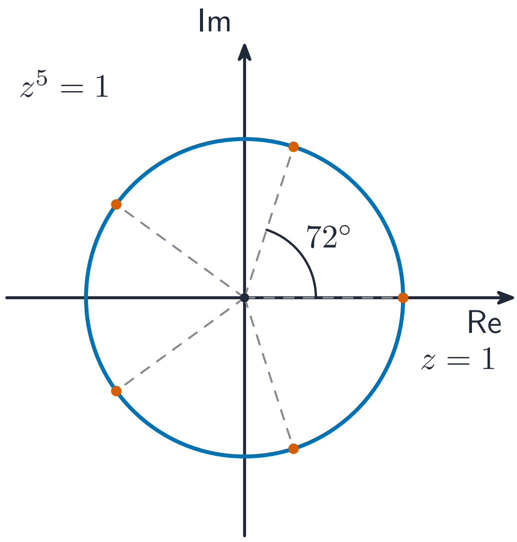
Build the skills to answer exam questions on **complex numbers 复数 — de Moivre's theorem 棣莫弗定理**, expanding $\cos n\theta/\sin n\theta$, and the **roots of unity 单位根**.

You must be able to:

- use $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$
- express $\cos n\theta/\sin n\theta$ in powers (and powers in multiple angles)
- find the nth roots of a complex number

1 Worked examples

■ De Moivre & roots of unity



The solutions of $z^n = 1$ are equally spaced around the unit circle

$\cos 3\theta$**:** real part of $(\cos\theta + i\sin\theta)^3 = \cos^3\theta - 3\cos\theta\sin^2\theta = 4\cos^3\theta - 3\cos\theta$.

Roots of unity: $z^n = 1 \Rightarrow z = e^{2\pi ik/n}$, $k = 0, \dots, n-1$.

2 Practice

2.1 Use de Moivre to write $(\cos\theta + i\sin\theta)^4$ in the form $\cos(\,) + i\sin(\,)$. [1]

2.2 Write down the three cube roots of unity in polar form. [2]

3 Exam-style questions

3.1 By de Moivre's theorem, $(\cos\theta + i\sin\theta)^5$ equals: [1]

- **A** $\cos 5\theta + i\sin 5\theta$
- **B** $5\cos\theta + 5i\sin\theta$
- **C** $\cos\theta + i\sin 5\theta$
- **D** $\cos^5\theta + i\sin^5\theta$

3.2 (a) Use de Moivre's theorem to show that $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$. [4]

(b) Hence solve $4\sin^3\theta - 3\sin\theta + \frac{1}{2} = 0$ for $0 < \theta < \frac{\pi}{2}$. [3]

3.3 Find the four roots of $z^4 = 16$, giving them in the form $x + iy$. [4]

Solutions

2.1 $\cos 4\theta + i\sin 4\theta$.

2.2 $1,\ \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3},\ \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}$.

3.1 **A**. $(\cos\theta + i\sin\theta)^5 = \cos 5\theta + i\sin 5\theta$.

3.2 (a) imaginary part of $(\cos\theta + i\sin\theta)^3 = \cos 3\theta + i\sin 3\theta$: $\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$; $= 3(1-\sin^2\theta)\sin\theta - \sin^3\theta = 3\sin\theta - 4\sin^3\theta$.
(b) $4\sin^3\theta - 3\sin\theta = -\sin 3\theta$, so the equation is $-\sin 3\theta + \frac{1}{2} = 0 \Rightarrow \sin 3\theta = \frac{1}{2}$; for $0 < 3\theta < \frac{3\pi}{2}$, $3\theta = \frac{\pi}{6}$ or $\frac{5\pi}{6}$; $\theta = \frac{\pi}{18}$ or $\frac{5\pi}{18}$.

3.3 $z^4 = 16 \Rightarrow z = 2e^{ik\pi/2}$, $k = 0, 1, 2, 3$; $z = 2,\ 2i,\ -2,\ -2i$.

8 **(a)** State the sum of the series $z + z^2 + ... + z^n$, for $z \neq 1$. [1]

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(b) By letting $z = \frac{1}{2}(\cos\theta + i\sin\theta)$ use de Moivre's theorem to deduce that

$$\sum_{m=1}^{n}\left(\frac{1}{2}\right)^{m}\sin m\theta = \frac{\left(\frac{1}{2}\right)^{n+2}\sin n\theta - \left(\frac{1}{2}\right)^{n+1}\sin(n+1)\theta + \frac{1}{2}\sin\theta}{\frac{5}{4} - \cos\theta}.$$ [6]

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(c) Use the result in **(b)** to find $\sum_{m=1}^{n}\left(\frac{1}{2}\right)^m m\cos m\theta$ in terms of n and θ. [You do not need to simplify your answer.] [3]

(d) Hence find $\sum_{m=1}^{\infty}\left(\frac{1}{2}\right)^m m\cos m\theta$ in terms of $\cos\theta$. [You may assume that $\frac{n}{2^n} \to 0$ as $n \to \infty$.] [2]

Additional page

If you use the following lined page to complete the answer(s) to any question(s), the question number(s) must be clearly shown.

2 Find the roots of the equation $z^4 = 8 - 8\mathrm{i}\sqrt{3}$, giving your answers in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r > 0$ and $-\pi \leqslant \theta < \pi$. [5]

4 **(a)** By considering the binomial expansion of $\left(z+\frac{1}{z}\right)$, where $z=\cos\theta+\mathrm{i}\sin\theta$, use de Moivre's theorem to show that

$$\cos^6\theta = a\cos 6\theta + b\cos 4\theta + c\cos 2\theta + d,$$

where a, b, c and d are constants to be determined. [5]

(b) Find the exact value of $\int_0^{\frac{1}{2}\pi} \cos^6 2x \, dx$. [3]

3 **(a)** Use de Moivre's theorem to show that

$$\sin 5\theta = 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta.$$ [4]

(b) Hence obtain the roots of the equation

$$32x^5 - 40x^3 + 10x - \sqrt{3} = 0$$

in the form $\sin q\pi$, where q is a rational number. [4]

1 Find the roots of the equation $z^3 = 27 - 27\mathrm{i}$, giving your answers in the form $r\mathrm{e}^{\mathrm{i}\theta}$, where $r > 0$ and $-\pi \leqslant \theta < \pi$. [5]

3 By considering the binomial expansion of $\left(z-\frac{1}{z}\right)$, where $z=\cos\theta+\mathrm{i}\sin\theta$, use de Moivre's theorem to show that

$$\operatorname{cosec}^5\theta=\frac{a}{\sin 5\theta+b\sin 3\theta+c\sin\theta},$$

where a, b and c are integers to be determined. [6]

5 **(a)** Use de Moivre's theorem to show that

$$\sec 5\theta = \frac{\sec^5\theta}{5\sec^4\theta - 20\sec^2\theta + 16}.$$ [6]

(b) Hence, obtain the roots of the equation

$$\sqrt{3}x^5 - 10x^4 + 40x^2 - 32 = 0$$

in the form $\sec(q\pi)$, where q is rational. [4]

2.6 Differential equations

Name: ____________ Class: ________ Date: __________ **Total: 14 marks**

KEY WORDS general solution 通解 • integrating factor 积分因子 • auxiliary equation 辅助方程 • complementary fu
• particular integral 特积分

Objective

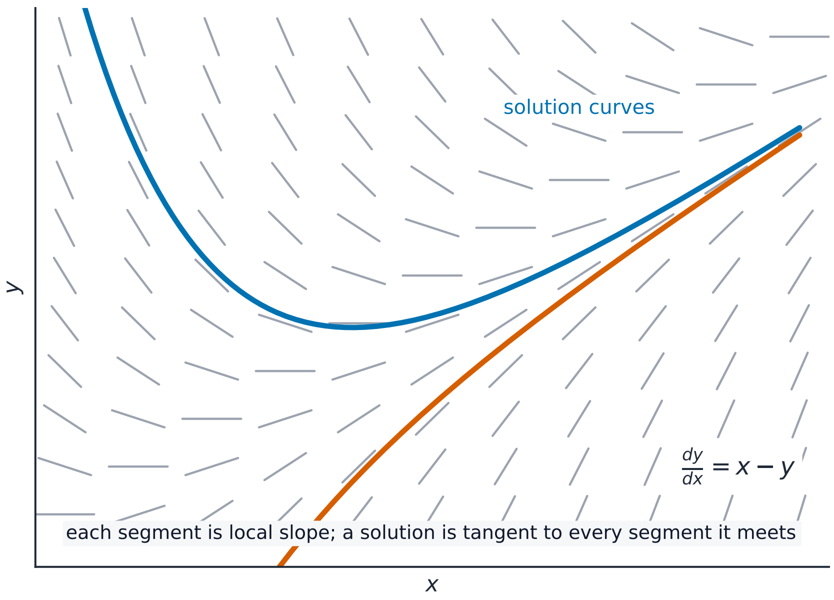
Build the skills to answer exam questions on **differential equations** 微分方程 — the **integrating factor** 积分因子 for first-order linear equations, and the **complementary function** 余函数 plus **particular integral** 特积分 for second-order constant-coefficient equations.

You must be able to:

- solve $\dfrac{dy}{dx} + P(x)y = Q(x)$ with $\mu = e^{\int P\,dx}$
- solve $ay'' + by' + cy = 0$ via the auxiliary equation (the CF)
- find a particular integral and the general solution

1 Worked examples

■ **Integrating factor & CF+PI**



A solution curve stays tangent to the slope field everywhere

$\dfrac{dy}{dx} + 2y = e^x$**:** $\mu = e^{2x}$; $\dfrac{d}{dx}(ye^{2x}) = e^{3x}$, so $y = \frac{1}{3}e^x + Ce^{-2x}$.

Second order: $y'' - 5y' + 6y = 0 \to$ auxiliary $m^2 - 5m + 6 = 0 \to y = Ae^{2x} + Be^{3x}$.

2 Practice

2.1 Write down the integrating factor for $\dfrac{dy}{dx} + 3y = x$. [1]

2.2 Write the auxiliary equation of $y'' - y' - 6y = 0$. [1]

3 Exam-style questions

3.1 The complementary function of $y'' + 4y = 0$ is: [1]

- **A** $Ae^{2x} + Be^{-2x}$
- **B** $A\cos 2x + B\sin 2x$
- **C** Ae^{2x}
- **D** $A + Bx$

3.2 Solve $\dfrac{dy}{dx} + \dfrac{1}{x}y = x^2$ (for $x > 0$), given $y = 1$ when $x = 1$. [5]

3.3 Solve $\dfrac{d^2y}{dx^2} - 3\dfrac{dy}{dx} + 2y = 4x$. [6]

Solutions

2.1 $\mu = e^{\int 3\,dx} = e^{3x}$.

2.2 $m^2 - m - 6 = 0$.

3.1 **B**. Auxiliary $m^2 + 4 = 0 \Rightarrow m = \pm 2i$, giving $A\cos 2x + B\sin 2x$.

3.2 $\mu = e^{\int (1/x)\,dx} = e^{\ln x} = x$; $\dfrac{d}{dx}(xy) = x^3$; $xy = \frac{1}{4}x^4 + C$; $y = 1$ at $x = 1$: $1 = \frac{1}{4} + C \Rightarrow C = \frac{3}{4}$; $y = \frac{1}{4}x^3 + \dfrac{3}{4x}$.

3.3 auxiliary $m^2 - 3m + 2 = 0 \Rightarrow m = 1, 2$, so CF $= Ae^x + Be^{2x}$; try PI $y = ax + b$: $-3a + 2(ax + b) = 4x \Rightarrow 2a = 4,\ -3a + 2b = 0$; $a = 2$, $b = 3$; general solution $y = Ae^x + Be^{2x} + 2x + 3$.

4 Find the particular solution of the differential equation

$$5\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = x^2 + 5x + 3,$$

given that, when $x = 0$, $y = \frac{dy}{dx} = 0$. [10]

7 **(a)** Show that $\frac{d}{dx}(\tanh^{-1}x) = \frac{1}{1-x^2}$. [3]

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(b) Find the solution of the differential equation

$$x\frac{dy}{dx} - y = x^2 \tanh^{-1}x,$$

for $0 < x < 1$, given that $y = 0$ when $x = \frac{1}{2}$. Give your answer in an exact form. [9]

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5 **(a)** Find the general solution of the differential equation

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 3y = 5\cos x. \qquad [7]$$

(b) For large positive values of x and for any initial conditions, show that the solution to part **(a)** can be approximated by

$$y \approx R\sin(x+\phi),$$

where the constants R and ϕ are to be determined. [3]

5 **(a)** Find the general solution of the differential equation

$$\frac{d^2y}{dx^2}+\frac{dy}{dx}+y=\sin 2x+\cos 2x. \qquad [7]$$

(b) For large positive values of x and for any initial conditions, show that the solution to part **(a)** can be approximated by

$$y \backsimeq R\sin(2x-\phi),$$

where R and ϕ are positive constants to be determined. [3]

6 Find the particular solution of the differential equation

$$\frac{dy}{dx} + \frac{1+x}{1-2x-x^2}y = 1$$

given that $y = \pi$ when $x = 0$. Give your answer in an exact form. [10]

5 Find the particular solution of the differential equation

$$6\frac{d^2x}{dt^2} + 3\frac{dx}{dt} + 6x = e^{-t},$$

given that, when $t = 0$, $x = \frac{dx}{dt} = 0$. [10]

7 Find the solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{x+5}{x^2+10x+61}y = 1,$$

given that $y = 0$ when $x = 3$. Give your answer in an exact form. [10]

4 Find the particular solution of the differential equation

$$\frac{d^2x}{dt^2} + \frac{dx}{dt} - 2x = 2t^2 + t - 1,$$

given that, when $t = 0$, $x = \frac{dx}{dt} = 0$. [10]

7 Find the solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} - \frac{2x+6}{x^2+6x+5}y = 4,$$

given that $y = 0$ when $x = 0$. Give your answer in an exact form. [9]