Exercise walk-through

Probability generating functions ▪ 4.5

eXercise

You must be able to

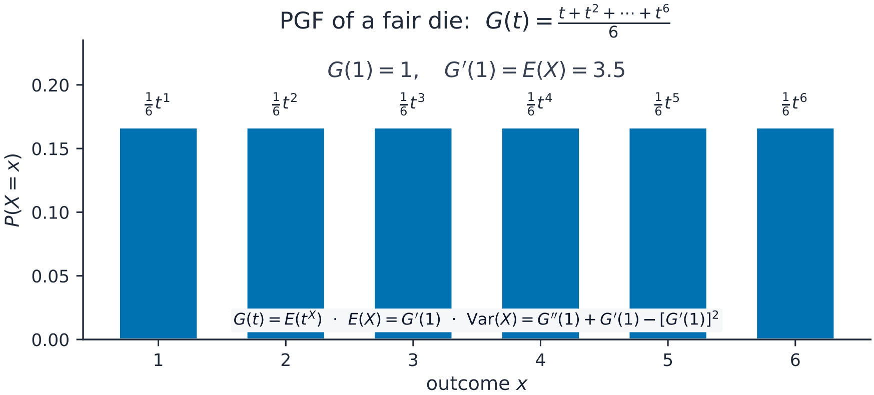
- write $G(t) = E(t^X) = \sum P(X = x)t^x$
- use $E(X) = G'(1)$ and $\mathrm{Var}(X) = G''(1) + G'(1) - [G'(1)]^2$
- multiply PGFs for a sum of independent variables

Method — Mean from the PGF

$P(X=0)=0.5$, $P(X=1)=0.3$, $P(X=2)=0.2$: $G(t)=0.5+0.3t+0.2t^2$;
$G'(t)=0.3+0.4t$, so $E(X)=G'(1)=0.7$.

An assortment of dice — discrete random variables

Method — Mean from the PGF



Each bar's probability is the coefficient of \$t^x\$ in \$G(t)\$; \$G'(1)=E(X)\$

Practice 2.1 ▪ 2 marks

Write down $G(t)$ for a variable with $P(X = 1) = P(X = 2) = P(X = 3) = \frac{1}{3}$.

Solution 2.1

Worked steps

$G(t) = \frac{1}{3}(t + t^2 + t^3)$ **M1** **A1**.

Practice 2.2 ▪ 1 mark

State the formula for $E(X)$ in terms of G.

Solution 2.2

Worked steps

$E(X) = G'(1)$ **B1**.

Exam-style 3.1 ▪ 1 mark

For any PGF, $G(1)$ equals:

A 0

B $E(X)$

C 1

D $\mathrm{Var}(X)$

Solution 3.1

A 0

B $E(X)$

C 1 ✓

D $\mathrm{Var}(X)$

Worked steps

C. $G(1) = \sum P(X = x) = 1.$

Exam-style 3.2 ▪ 3 marks

A variable has $G(t) = \frac{1}{4}(1 + t)^2$.
(a) Find the probability distribution of X.
(b) Find $E(X)$ and $\mathrm{Var}(X)$.

Solution 3.2

Method: Mean from the PGF

Worked steps

(a) $G(t) = \frac{1}{4}(1 + 2t + t^2) = \frac{1}{4} + \frac{1}{2}t + \frac{1}{4}t^2$, so $P(X = 0) = \frac{1}{4}$, $P(X = 1) = \frac{1}{2}$, $P(X = 2) = \frac{1}{4}$ **M1** **M1** **A1**.

(b) $G'(t) = \frac{1}{2}+\frac{1}{2}t \Rightarrow E(X) = G'(1) = 1$ **M1**; $G''(t) = \frac{1}{2} \Rightarrow \mathrm{Var}(X) = \frac{1}{2}+1-1^2 = \frac{1}{2}$ **M1** **M1** **A1**.

Exam-style 3.3 ▪ 3 marks

X and Y are independent with PGFs $G_X(t) = \frac{1}{2}(1+t)$ and $G_Y(t) = \frac{1}{3}(1+t+t^2)$. Find the PGF of $X+Y$ and hence $P(X+Y=0)$.

Solution 3.3

Worked steps

$G_{X+Y}(t) = G_X(t)G_Y(t) = \frac{1}{2}(1+t) \cdot \frac{1}{3}(1+t+t^2) = \frac{1}{6}(1+t)(1+t+t^2)$ **M1** **M1**;
$P(X+Y=0)$ is the constant term $= \frac{1}{6}$ **A1**.