Exercise walk-through

Complex numbers ▪ 2.5

eXercise

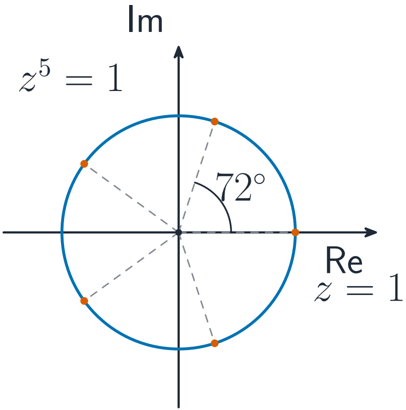
You must be able to

- use $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$
- express $\cos n\theta/\sin n\theta$ in powers (and powers in multiple angles)
- find the nth roots of a complex number

Method — De Moivre & roots of unity

$\cos 3\theta$: real part of $(\cos\theta + i\sin\theta)^3 = \cos^3\theta - 3\cos\theta\sin^2\theta = 4\cos^3\theta - 3\cos\theta$.
Roots of unity: $z^n = 1 \Rightarrow z = e^{2\pi ik/n}$, $k = 0, \ldots, n-1$.

Method — De Moivre & roots of unity



Five points evenly spaced around the unit circle

Practice 2.1 ▪ 1 mark

Use de Moivre to write $(\cos\theta + i\sin\theta)^4$ in the form $\cos(\ ) + i\sin(\ )$.

Solution 2.1

Method: De Moivre & roots of unity

Worked steps

$\cos 4\theta + i \sin 4\theta$ **B1**.

Practice 2.2 ▪ 2 marks

Write down the three cube roots of unity in polar form.

Solution 2.2

Method: De Moivre & roots of unity

Worked steps

$1,\ \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3},\ \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}$ **B1** **B1**.

Exam-style 3.1 ▪ 1 mark

By de Moivre's theorem, $(\cos\theta + i\sin\theta)^5$ equals:

A $\cos 5\theta + i\sin 5\theta$

B $5\cos\theta + 5i\sin\theta$

C $\cos\theta + i\sin 5\theta$

D $\cos^5\theta + i\sin^5\theta$

Solution 3.1

Method: De Moivre & roots of unity

A $\cos 5\theta + i \sin 5\theta$ ✓

B $5 \cos \theta + 5i \sin \theta$

C $\cos \theta + i \sin 5\theta$

D $\cos^5 \theta + i \sin^5 \theta$

Worked steps

A. $(\cos \theta + i \sin \theta)^5 = \cos 5\theta + i \sin 5\theta$.

Exam-style 3.2 ▪ 4 marks

(a) Use de Moivre's theorem to show that $\sin 3\theta = 3\sin\theta - 4\sin^3\theta$.
(b) Hence solve $4\sin^3\theta - 3\sin\theta + \frac{1}{2} = 0$ for $0 < \theta < \frac{\pi}{2}$.

Solution 3.2

Method: De Moivre & roots of unity

Worked steps

(a) imaginary part of $(\cos\theta + i\sin\theta)^3 = \cos 3\theta + i\sin 3\theta$: $\sin 3\theta = 3\cos^2\theta\sin\theta - \sin^3\theta$ M1 M1; $= 3(1-\sin^2\theta)\sin\theta - \sin^3\theta = 3\sin\theta - 4\sin^3\theta$ M1 A1.
(b) $4\sin^3\theta - 3\sin\theta = -\sin 3\theta$, so the equation is $-\sin 3\theta + \frac{1}{2} = 0 \Rightarrow \sin 3\theta = \frac{1}{2}$ M1; for $0 < 3\theta < \frac{3\pi}{2}$, $3\theta = \frac{\pi}{6}$ or $\frac{5\pi}{6}$ M1; $\theta = \frac{\pi}{18}$ or $\frac{5\pi}{18}$ A1.

Exam-style 3.3 ▪ 4 marks

Find the four roots of $z^4 = 16$, giving them in the form $x + iy$.

Solution 3.3

Method: De Moivre & roots of unity

Worked steps

$z^4 = 16 \Rightarrow z = 2e^{ik\pi/2},\ k = 0, 1, 2, 3$ **M1** **M1**; $z = 2,\ 2i,\ -2,\ -2i$ **A1** **A1**.