Exercise walk-through

Matrices ▪ 2.2

eXercise

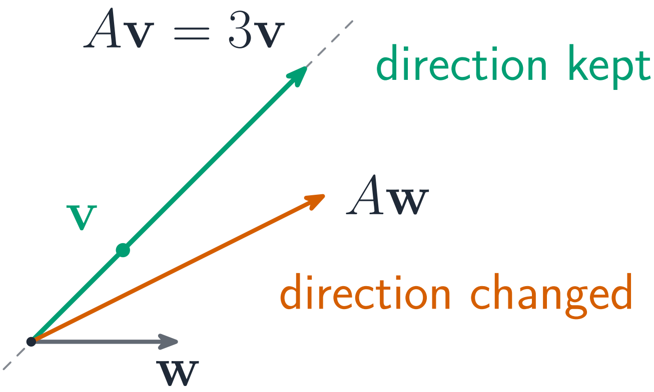
You must be able to

- find eigenvalues from $\det(A - \lambda I) = 0$
- find an eigenvector for each eigenvalue
- write $A = QDQ^{-1}$ and use it (e.g. for A^n)

Method — Eigenvalues & eigenvectors

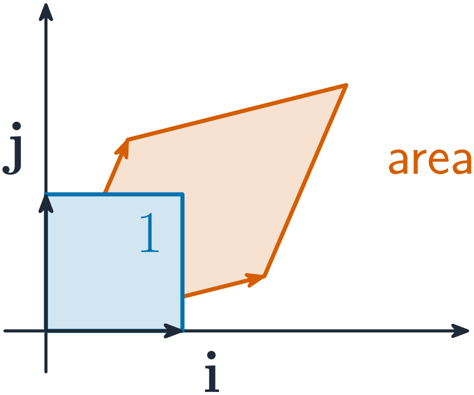
$A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$: $\det(A - \lambda I) = (2-\lambda)^2 - 1 = 0 \Rightarrow \lambda = 1$ or 3. Eigenvector for $\lambda = 3$: $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$; for $\lambda = 1$: $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.

Method — Eigenvalues & eigenvectors



An eigenvector stretched along its own line; a general vector turned aside

Method — Eigenvalues & eigenvectors



area $= \det = 2$

Eigenvectors are the directions a matrix leaves unturned

Practice 2.1 ▪ 1 mark

Write down the characteristic equation of a matrix A.

Solution 2.1

Worked steps

$\det(A - \lambda I) = 0$ **B1**.

Practice 2.2 ▪ 1 mark

Find the eigenvalues of $\begin{pmatrix} 4 & 0 \\ 0 & -2 \end{pmatrix}$.

Solution 2.2

Method: Eigenvalues & eigenvectors

Worked steps

$\lambda = 4$ and $\lambda = -2$ (the diagonal entries) **B1**.

Exam-style 3.1 ▪ 1 mark

A vector $\mathbf{v}$ is an eigenvector of A with eigenvalue λ when:

A $A\mathbf{v} = \mathbf{0}$

B $A\mathbf{v} = \lambda\mathbf{v}$

C $\det A = \lambda$

D $A = \lambda I$

Solution 3.1

Method: Eigenvalues & eigenvectors

A $A\mathbf{v} = \mathbf{0}$

B $A\mathbf{v} = \lambda\mathbf{v}$ ✓

C $\det A = \lambda$

D $A = \lambda I$

Worked steps

B. $A\mathbf{v} = \lambda\mathbf{v}$ defines an eigenvector.

Exam-style 3.2 ▪ 2 marks

The matrix $A = \begin{pmatrix} 3 & 1 \\ 0 & 2 \end{pmatrix}$.

(a) Find the eigenvalues.

(b) Find an eigenvector for each eigenvalue.

Solution 3.2

Method: Eigenvalues & eigenvectors

Worked steps

(a) $\det\begin{pmatrix}3-\lambda & 1 \\ 0 & 2-\lambda\end{pmatrix} = (3-\lambda)(2-\lambda) = 0 \Rightarrow \lambda = 3, 2$ **M1** **A1**.

(b) $\lambda = 3$: $(A - 3I)\mathbf{v} = 0 \Rightarrow \begin{pmatrix}0 & 1 \\ 0 & -1\end{pmatrix}\mathbf{v} = 0 \Rightarrow \mathbf{v} = \begin{pmatrix}1 \\ 0\end{pmatrix}$ **M1** **A1**; $\lambda = 2$: $\begin{pmatrix}1 & 1 \\ 0 & 0\end{pmatrix}\mathbf{v} = 0 \Rightarrow \mathbf{v} = \begin{pmatrix}1 \\ -1\end{pmatrix}$ **M1** **A1**.

Exam-style 3.3 ▪ 5 marks

The matrix $M = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$ has eigenvalues 1 and 3. Using $M = QDQ^{-1}$, find M^2 and verify by direct multiplication.

Solution 3.3

Method: Eigenvalues & eigenvectors

Worked steps

$D = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$, so $M^2 = QD^2Q^{-1}$ with $D^2 = \begin{pmatrix} 1 & 0 \\ 0 & 9 \end{pmatrix}$ M1 M1; this gives $M^2 = \begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix}$ M1; direct: $\begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}^2 = \begin{pmatrix} 5 & 4 \\ 4 & 5 \end{pmatrix}$ M1 A1.