Past-paper walk-through

Proof by induction ▪ 1.7

eXercise

Questions in this set

- 9231/11 N25 Q3 ▪ 7 marks
- 9231/12 N25 Q3 ▪ 8 marks
- 9231/14 N25 Q1 ▪ 5 marks
- 9231/11 J25 Q3 ▪ 7 marks
- 9231/13 J25 Q2 ▪ 6 marks

Reading the mark scheme

- **B1** a mark that stands on its own – the point is either made or not.
- **M1** a *method* mark: the right approach, even if the number is wrong.
- **A1** an *answer* mark, and it usually needs its M mark first.
- **C1** a working mark on the way to the answer.
- // separates answers the examiner accepts equally.
- **ecf** = error carried forward: a later part is marked on your own earlier answer.
- **owtte** = "or words to that effect" – the wording need not match.

Question 3

9231/11 N25 ▪ Q3 ▪ 7 marks

Prove by mathematical induction that, for every positive integer n,

$$\frac{d^{2n-1}}{dx^{2n-1}}(x\cos x) = (-1)^n(x\sin x - (2n-1)\cos x).$$ **[7]**

Solution 3

Mark scheme

- Base $n = 1$: $\frac{d}{dx}(x\cos x) = \cos x - x\sin x = (-1)^1(x\sin x - \cos x)$. Assume true for $n = k$; differentiating the hypothesis twice gives $\frac{d^{2k+1}}{dx^{2k+1}}(x\cos x) = (-1)^{k+1}(x\sin x - (2k+1)\cos x)$ — the $n = k + 1$ case. Hence true for all positive integers n by induction.

Question 3

9231/12 N25 ▪ Q3 ▪ 8 marks

The sequence of positive numbers $u_1, u_2, u_3, \ldots$ is such that $u_1 < 5$ and, for $n \geqslant 1$,

$$u_{n+1} = \frac{6u_n + 5}{u_n + 2}.$$

(a) By considering $5 - u_{n+1}$, prove by mathematical induction that $u_n < 5$ for all positive integers n. **[5]**

(b) Show that $u_{n+1} > u_n$ for $n \geqslant 1$. **[3]**

Solution 3

(a)

Mark scheme

- Base: $u_1 < 5$. Step: $5 - u_{k+1} = 5 - \dfrac{6u_k + 5}{u_k + 2} = \dfrac{5 - u_k}{u_k + 2} > 0$ when $u_k < 5$ (and $u_k > 0$), so $u_{k+1} < 5$. By induction $u_n < 5$ for all n.

Solution 3

(b)

Mark scheme

- $u_{n+1} - u_n = \dfrac{6u_n + 5}{u_n + 2} - u_n = \dfrac{-(u_n - 5)(u_n + 1)}{u_n + 2}$, which is > 0 since $0 < u_n < 5$. Hence $u_{n+1} > u_n$.

Question 1

9231/14 N25 ▪ Q1 ▪ 5 marks

Prove by mathematical induction that $\left(\frac{6}{5}\right)^n \geqslant 1 + \frac{1}{5}n$ for every positive integer n. **[5]**

Solution 1

Mark scheme

- Base case $n = 1$: LHS $= \left(\frac{6}{5}\right)^1 = \frac{6}{5}$, RHS $= 1 + \frac{1}{5} = \frac{6}{5}$. Assume $\left(\frac{6}{5}\right)^k \geqslant 1 + \frac{1}{5}k$ for some positive integer k. Then $\left(\frac{6}{5}\right)^{k+1} \geqslant \frac{6}{5}\left(1 + \frac{1}{5}k\right) = \frac{6}{5} + \frac{6}{25}k \geqslant \frac{6}{5} + \frac{1}{5}k = 1 + \frac{1}{5}(k+1)$. Hence by induction $\left(\frac{6}{5}\right)^n \geqslant 1 + \frac{1}{5}n$ for every positive integer n.

Question 3

9231/11 J25 ▪ Q3 ▪ 7 marks

The sequence $u_1, u_2, u_3, \ldots$ is such that $u_1 = 5$ and $u_{n+1} = 6u_n + 5$ for $n \geqslant 1$.

(a) Prove by induction that $u_n = 6^n - 1$ for all positive integers n. **[5]**

(b) Deduce that u_{2n} is divisible by u_n for $n \geqslant 1$. **[2]**

Solution 3

(a)

Mark scheme

- Base case: $u_1 = 5 = 6^1 - 1$. Assume $u_k = 6^k - 1$. Then $u_{k+1} = 6(6^k - 1) + 5 = 6^{k+1} - 6 + 5 = 6^{k+1} - 1$, the $n = k + 1$ case. Hence by induction $u_n = 6^n - 1$ for all positive integers n.

Solution 3

(b)

Mark scheme

- $\dfrac{u_{2n}}{u_n} = \dfrac{6^{2n}-1}{6^n-1} = \dfrac{(6^n-1)(6^n+1)}{6^n-1} = 6^n+1$, an integer. Hence u_{2n} is divisible by u_n.

Question 2

9231/13 J25 ▪ Q2 ▪ 6 marks

Prove by mathematical induction that $2025^n + 47^n - 2$ is divisible by 46 for all positive integers n. **[6]**

Solution 2

Mark scheme

- Base case $n = 1$: $2025 + 47 - 2 = 2070 = 45 \times 46$, divisible by 46 . Assume $2025^k + 47^k - 2$ is divisible by 46. Then $2025^{k+1} + 47^{k+1} - 2 = (2024 + 1)2025^k + (46 + 1)47^k - 2 = (2025^k + 47^k - 2) + 2024 \times 2025^k + 46 \times 47^k$, where $2024 \times 2025^k + 46 \times 47^k = 46(44 \times 2025^k + 47^k)$ is divisible by 46. Hence by induction it is true for every positive integer n.