

A-Level Further Mathematics

Every topic handout in one volume

exercises.lol

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Further Pure Mathematics 1

A-Level Further Mathematics

This handout covers Topic 1: **Further Pure Mathematics**. It adds new algebra, matrices, polar coordinates, more vectors, and proof by induction.

Roots of polynomial equations

For a polynomial equation, the **roots** are linked to the **coefficients**. For $ax^2 + bx + c = 0$ with roots α, β :

$$\alpha + \beta = -\frac{b}{a}, \quad \alpha\beta = \frac{c}{a}.$$

For a cubic $ax^3 + bx^2 + cx + d = 0$ with roots α, β, γ :

$$\sum \alpha = -\frac{b}{a}, \quad \sum \alpha\beta = \frac{c}{a}, \quad \alpha\beta\gamma = -\frac{d}{a}.$$

Sums like $\sum \alpha$ and $\sum \alpha\beta$ are **symmetric functions** of the roots (unchanged if the roots are swapped). To find an equation whose roots are changed in a simple way, use a **substitution** (for example, put $w = \alpha + 1$).

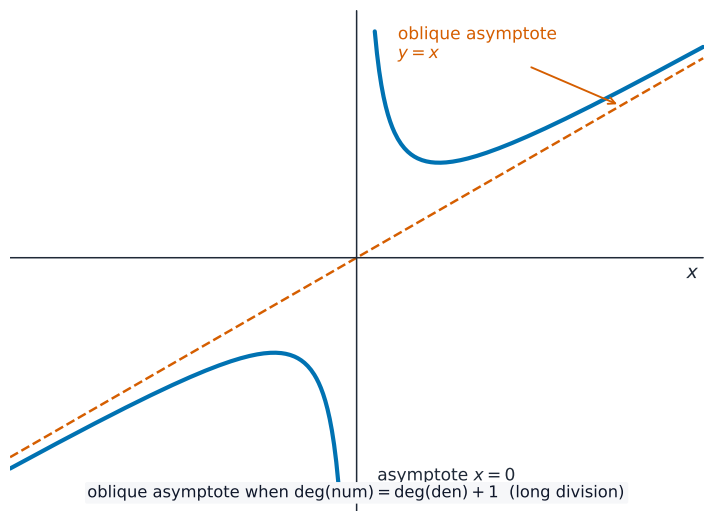
Worked example. The equation $x^2 - 5x + 6 = 0$ has roots α, β . Find the equation with roots $\alpha + 1, \beta + 1$.

Here $\alpha + \beta = 5$ and $\alpha\beta = 6$. The new sum is $(\alpha + 1) + (\beta + 1) = 7$, and the new product is $(\alpha + 1)(\beta + 1) = \alpha\beta + \alpha + \beta + 1 = 6 + 5 + 1 = 12$. So the new equation is

$$x^2 - 7x + 12 = 0.$$

Rational functions and graphs

A **rational function** is a fraction of two polynomials. When the top has degree exactly one higher than the bottom, the graph has an **oblique asymptote** (a slanted line the curve approaches); find it by dividing out. You should also be able to relate the graph of $y = f(x)$ to those of $y^2 = f(x)$, $y = \frac{1}{f(x)}$, $y = |f(x)|$ and $y = f(|x|)$. Find the set of values the function can take using the **discriminant**, and locate its **turning points**.



Far from the origin the curve hugs the slant line $y = x$; near $x = 0$ it runs off along the vertical asymptote.

Worked example. Find the oblique asymptote of $y = \frac{x^2 + x + 1}{x + 1}$.

Divide out the fraction: $\frac{x^2 + x + 1}{x + 1} = x + \frac{1}{x + 1}$. As $x \rightarrow \pm\infty$ the remainder $\frac{1}{x + 1} \rightarrow 0$, so the curve approaches the line $y = x$ — the oblique asymptote.

Summation of series

Learn the standard results:

$$\sum_{r=1}^n r = \frac{1}{2}n(n+1), \quad \sum_{r=1}^n r^2 = \frac{1}{6}n(n+1)(2n+1), \quad \sum_{r=1}^n r^3 = \frac{1}{4}n^2(n+1)^2.$$

The **method of differences** sums a series by cancelling middle terms: if **partial fractions** write each term as $f(r) - f(r+1)$, almost everything cancels. From the sum to n terms you can see whether a series is **convergent** and, if so, find its **sum to infinity**.

Worked example. Find $\sum_{r=1}^n (2r + 1)$.

$$\sum_{r=1}^n (2r + 1) = 2 \sum_{r=1}^n r + \sum_{r=1}^n 1 = 2 \cdot \frac{1}{2}n(n+1) + n = n(n+1) + n = n(n+2).$$

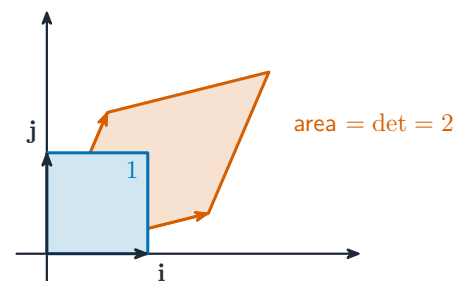
Matrices

A **matrix** is a rectangular block of numbers. **Matrix addition** adds entries in matching positions; you can also subtract and multiply matrices (multiplication is row $\times$ column). The **zero matrix** has every entry 0, and the **identity matrix** (or **unit matrix**) I leaves any matrix unchanged under multiplication.

For a 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, the **determinant** is $\det A = ad - bc$. If $\det A \neq 0$ the matrix is non-singular (otherwise it is a **singular matrix**), and the **inverse matrix** is

$$A^{-1} = \frac{1}{ad - bc} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

For a product, $(AB)^{-1} = B^{-1}A^{-1}$. A 2×2 matrix can represent a **geometric transformation** of the plane — a **rotation**, **reflection**, **enlargement**, **stretch** or **shear**: the determinant gives the area scale factor, and a product AB means "do B , then A ". Points or lines that do not move are called **invariant points** and **invariant lines**.



The matrix sends the unit square to a parallelogram whose area is the determinant.

Worked example. Find the inverse of $A = \begin{pmatrix} 3 & 1 \\ 2 & 4 \end{pmatrix}$.

$\det A = 3 \times 4 - 1 \times 2 = 10$, so

$$A^{-1} = \frac{1}{10} \begin{pmatrix} 4 & -1 \\ -2 & 3 \end{pmatrix}.$$

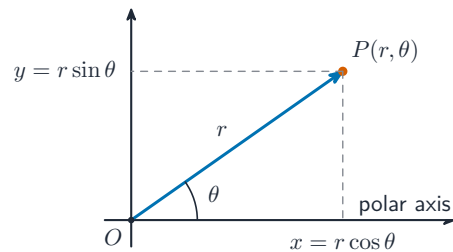
Polar coordinates



A spiral staircase: spirals are described naturally using polar coordinates.

Polar coordinates give a point by its distance r from the origin and its angle θ . They link to **Cartesian coordinates** by

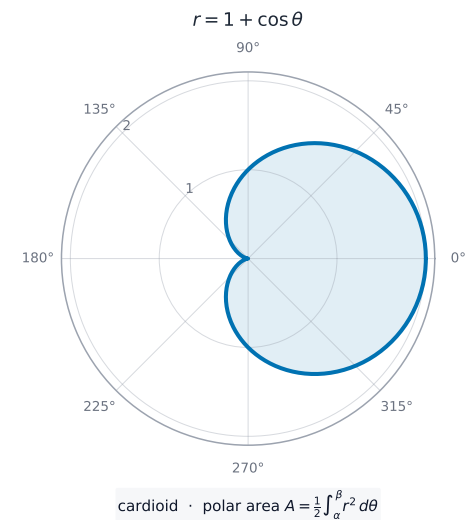
$$x = r \cos \theta, \quad y = r \sin \theta, \quad r^2 = x^2 + y^2.$$



Polar coordinates (r, θ) convert to Cartesian by $x = r \cos \theta$ and $y = r \sin \theta$.

You should sketch simple **polar curves**, and find the area of a sector with

$$\text{area} = \frac{1}{2} \int r^2 d\theta.$$



The cardioid $r = 1 + \cos \theta$, plotted straight from r as a function of θ .

Worked example. Convert the polar equation $r = 4 \cos \theta$ to Cartesian form.

Multiply both sides by r : $r^2 = 4r \cos \theta$, so $x^2 + y^2 = 4x$. Completing the square gives $(x - 2)^2 + y^2 = 4$, a circle of radius 2 centred at $(2, 0)$.

Vectors

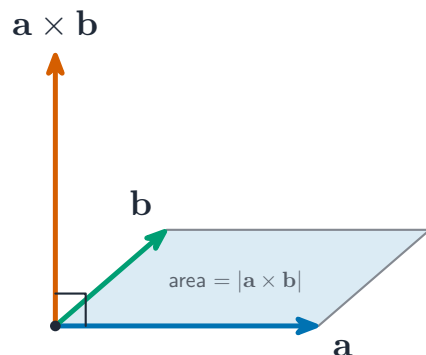


Forces like wind and water are vectors — they have both size and direction.

In three dimensions a **plane** can be written as $ax + by + cz = d$, or $\mathbf{r} \cdot \mathbf{n} = p$, or $\mathbf{r} = \mathbf{a} + \lambda\mathbf{b} + \mu\mathbf{c}$. Besides the scalar product, there is the **vector product** of two **vectors** :

$$\mathbf{a} \times \mathbf{b} = |\mathbf{a}| |\mathbf{b}| \sin \theta \hat{\mathbf{n}},$$

which gives a vector at right angles to both. With scalar and vector products you can find distances, angles, and where lines and planes meet — including the shortest distance between **skew lines** .



$\mathbf{a} \times \mathbf{b}$ is perpendicular to both vectors; its length equals the area of the parallelogram they span.

Worked example. Find $\mathbf{a} \times \mathbf{b}$ for $\mathbf{a} = \mathbf{i} + \mathbf{j}$ and $\mathbf{b} = \mathbf{j} + \mathbf{k}$.

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} = \mathbf{i}(1) - \mathbf{j}(1) + \mathbf{k}(1) = \mathbf{i} - \mathbf{j} + \mathbf{k}.$$

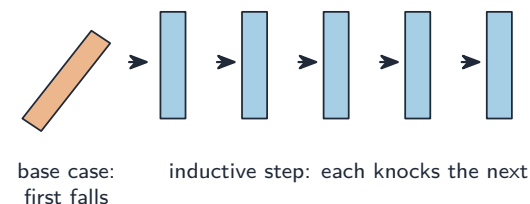
Proof by induction

Mathematical induction proves a result for every positive integer n in two steps:

1. **Base case:** show the result is true for $n = 1$.
2. **Inductive step:** assume it is true for $n = k$, then prove it for $n = k + 1$.

If both steps work, the result is true for all n . Often you first make a **conjecture** (a sensible guess) from a few cases, then confirm it by **inductive proof** .

so every domino falls — true for all n



Induction is like dominoes: the base case topples the first, and the inductive step makes each one knock over the next — so the result holds for every n

Worked example. Prove that $\sum_{r=1}^n r = \frac{1}{2}n(n+1)$.

Base case $n = 1$: the left side is 1 and the right side is $\frac{1}{2}(1)(2) = 1$. True. **Inductive**

step: assume $\sum_{r=1}^k r = \frac{1}{2}k(k+1)$. Then

$$\sum_{r=1}^{k+1} r = \frac{1}{2}k(k+1) + (k+1) = (k+1) \left(\frac{k}{2} + 1 \right) = \frac{1}{2}(k+1)(k+2).$$

This is the formula with $n = k + 1$, so by induction it holds for all n .

Exam tips

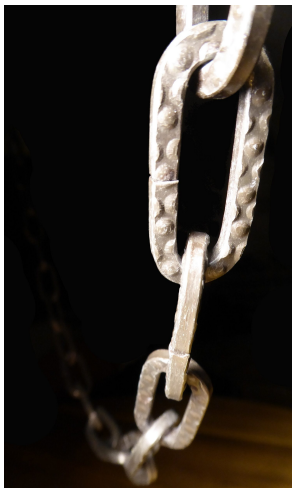
- Use the **relationships between roots and coefficients** (sum, sum of pairs, product) instead of solving the polynomial.
- For **proof by induction**, set out the base case, assume the result for $n = k$, prove it for $n = k + 1$, and write a clear closing statement.
- Sketch **rational-function graphs** with their asymptotes and turning points; use the **method of differences** for a summation.
- For **matrices**, know the determinant, inverse and invariant lines, and interpret each transformation geometrically.

Further Pure Mathematics 2

A-Level Further Mathematics

This handout covers Topic 2: **Further Pure Mathematics** 2. It adds hyperbolic functions, eigenvalues, new differentiation and integration, de Moivre's theorem, and methods for differential equations.

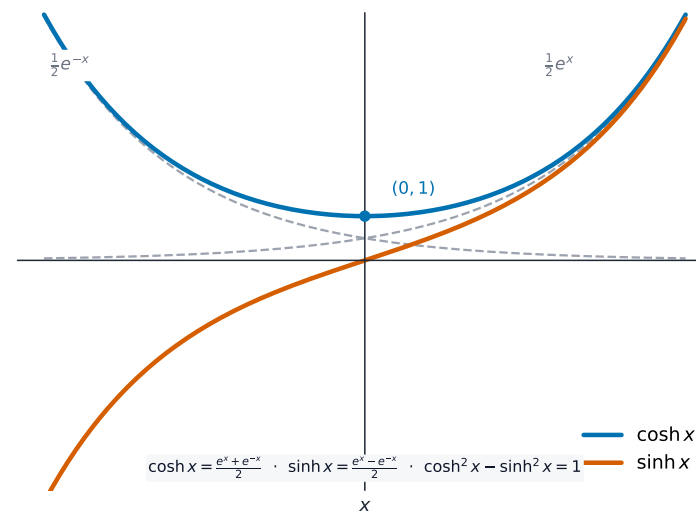
Hyperbolic functions



A hanging cable forms a catenary — the curve of the hyperbolic cosine.

The **hyperbolic functions** are built from the exponential function:

$$\cosh x = \frac{e^x + e^{-x}}{2}, \quad \sinh x = \frac{e^x - e^{-x}}{2}, \quad \tanh x = \frac{\sinh x}{\cosh x}.$$



$\cosh$ is the average of e^x and e^{-x} ; $\sinh$ is half their difference.

They obey identities much like the trigonometric ones, the main one being $\cosh^2 x - \sinh^2 x = 1$. The three **reciprocals** complete the set: $\operatorname{sech} x = \frac{1}{\cosh x}$, $\operatorname{cosech} x = \frac{1}{\sinh x}$, and $\coth x = \frac{1}{\tanh x} = \frac{\cosh x}{\sinh x}$ — all built from e^x too, and worth recognising in integrals and mark schemes. The **inverse hyperbolic functions** have a **logarithmic form**, for example $\sinh^{-1} x = \ln(x + \sqrt{x^2 + 1})$.

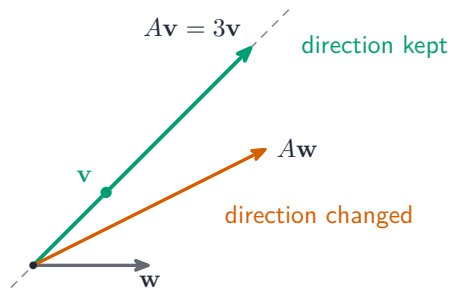
Worked example. Show that $\cosh^2 x - \sinh^2 x = 1$.

$$\cosh^2 x - \sinh^2 x = \frac{(e^x + e^{-x})^2}{4} - \frac{(e^x - e^{-x})^2}{4} = \frac{(e^{2x} + 2 + e^{-2x}) - (e^{2x} - 2 + e^{-2x})}{4} = \frac{4}{4} = 1.$$

Matrices

You can write three linear equations in three unknowns as a single **matrix equation** $A\mathbf{x} = \mathbf{b}$. If A is non-singular there is one solution; if A is singular the equations are either inconsistent (no solution) or have infinitely many.

For a square matrix A , the **characteristic equation** is $\det(A - \lambda I) = 0$. Its solutions are the **eigenvalues** λ , and for each one the vector $\mathbf{v}$ with $A\mathbf{v} = \lambda\mathbf{v}$ is an **eigenvector**. You can then write $A = QDQ^{-1}$, where D is a **diagonal matrix** of eigenvalues and the columns of Q are the eigenvectors.



Multiplying an eigenvector by A only stretches it; a general vector is turned to a new direction.

Worked example. Find the eigenvalues of $A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$.

The characteristic equation is

$$\det \begin{pmatrix} 2 - \lambda & 1 \\ 1 & 2 - \lambda \end{pmatrix} = (2 - \lambda)^2 - 1 = 0 \Rightarrow 2 - \lambda = \pm 1.$$

So $\lambda = 1$ or $\lambda = 3$. (The eigenvector for $\lambda = 3$ is $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and for $\lambda = 1$ is $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$.)

Differentiation

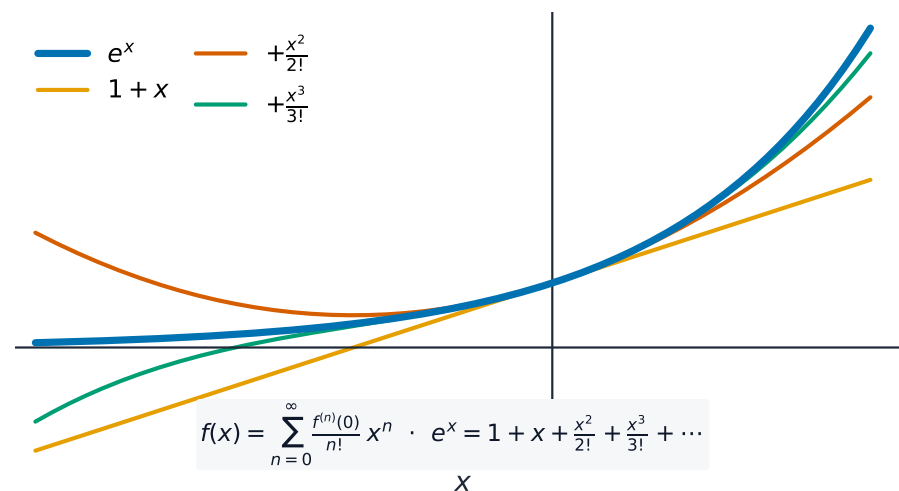
You can now differentiate the hyperbolic functions and the inverse functions $\sin^{-1} x$, $\tan^{-1} x$, $\sinh^{-1} x$ and so on. You can also find $\frac{d^2 y}{dx^2}$ for curves given implicitly or parametrically.

A **Maclaurin's series**

writes a function as a power series:

$$f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots$$

For example $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$



Adding more terms makes the polynomial match e^x over a wider stretch around $x = 0$.

Worked example. Find the Maclaurin series of $f(x) = \ln(1 + x)$ up to the term in x^3 .

$f(0) = 0$; $f'(x) = \frac{1}{1+x}$ so $f'(0) = 1$; $f''(x) = \frac{-1}{(1+x)^2}$ so $f''(0) = -1$; $f'''(x) = \frac{2}{(1+x)^3}$ so $f'''(0) = 2$. Substituting into the series:

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \dots$$

Integration

Learn these standard integrals:

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a} + C, \quad \int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a} + C.$$

A **trigonometric substitution**, a **hyperbolic substitution**, or **completing the square** in the denominator handles related forms. A **reduction formula** links an integral I_n to I_{n-1} , so you can work down step by step. Integration also gives the **arc length** of a curve and the **surface area of revolution** when a curve is turned about an axis.

Worked example. Find $\int \frac{1}{\sqrt{4-x^2}} dx$.

Here $a^2 = 4$, so $a = 2$ and

$$\int \frac{1}{\sqrt{4-x^2}} dx = \sin^{-1} \frac{x}{2} + C.$$

Bounding a sum by an integral. Draw rectangles of unit width under, or over, a decreasing curve such as $y = \frac{1}{x}$. Each rectangle of height $\frac{1}{r}$ is trapped between two integral strips, so summing gives

$$\ln(n+1) < \sum_{r=1}^n \frac{1}{r} < 1 + \ln n.$$

Shrinking the rectangle width to $\frac{1}{n} \rightarrow 0$ turns such a sum into a **Riemann sum** that equals the integral in the limit, e.g. $\lim_{n \rightarrow \infty} \frac{1}{n} \sum_{r=1}^n f\left(\frac{r}{n}\right) = \int_0^1 f(x) dx$.

Complex numbers

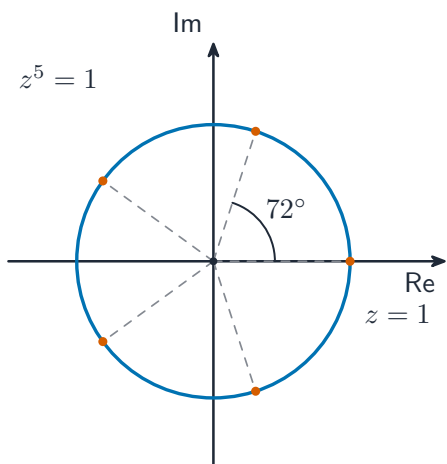


Self-similar patterns like Romanesco broccoli arise from iterating functions in the complex plane.

A **complex number** in polar form is $z = r(\cos \theta + i \sin \theta)$. **De Moivre's theorem** says that for any integer n ,

$$(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta.$$

It is used to expand $\cos n\theta$ and $\sin n\theta$ in powers, to sum series, and to find the **n roots of unity** (the solutions of $z^n = 1$, equally spaced around the unit circle).



The five solutions of $z^5 = 1$ are equally spaced 72° apart on the unit circle.

Worked example. Use de Moivre's theorem to express $\cos 3\theta$ in terms of $\cos \theta$.

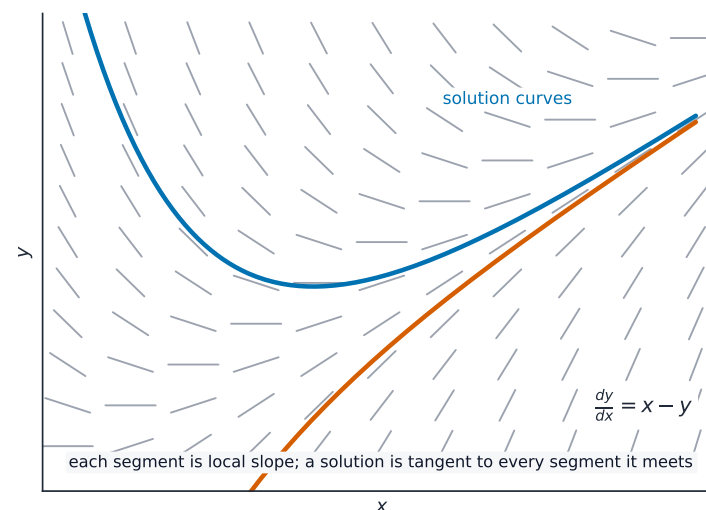
Take the real part of $(\cos \theta + i \sin \theta)^3 = \cos 3\theta + i \sin 3\theta$:

$$\cos 3\theta = \cos^3 \theta - 3 \cos \theta \sin^2 \theta = \cos^3 \theta - 3 \cos \theta (1 - \cos^2 \theta) = 4 \cos^3 \theta - 3 \cos \theta.$$

Differential equations

For a first order linear equation $\frac{dy}{dx} + P(x)y = Q(x)$, multiply by the **integrating factor** $\mu = e^{\int P dx}$. The left side then becomes $\frac{d}{dx}(\mu y)$, so you can integrate directly.

For a linear equation with constant coefficients, the **general solution** is the sum of two parts: the **complementary function** (the solution of the equation with the right side set to 0, found from the auxiliary equation) and a **particular integral** (any one solution of the full equation). **Initial conditions** then fix the constants to give the **particular solution**; simpler equations with **separable variables** are integrated directly.



A solution curve follows the slope field, staying tangent to the little segments everywhere.

Worked example. Solve $\frac{dy}{dx} + 2y = e^x$.

The integrating factor is $\mu = e^{\int 2 dx} = e^{2x}$. Multiplying through gives $\frac{d}{dx}(y e^{2x}) = e^{3x}$, so

$$y e^{2x} = \frac{1}{3} e^{3x} + C \Rightarrow y = \frac{1}{3} e^x + C e^{-2x}.$$

Second order, constant coefficients. For $\frac{d^2 y}{dx^2} + b \frac{dy}{dx} + cy = f(x)$, solve the **auxiliary equation** $m^2 + bm + c = 0$. The complementary function depends on

its roots:

- two distinct real roots m_1, m_2 : $y = Ae^{m_1x} + Be^{m_2x}$;
- a repeated root m : $y = (A + Bx)e^{mx}$;
- complex roots $p \pm qi$: $y = e^{px}(A \cos qx + B \sin qx)$.

Then add a particular integral by trying a form matching $f(x)$ (a polynomial, ae^{bx} , or $a \cos px + b \sin px$); if that trial already appears in the complementary function, **multiply it by x** .

Worked example. Solve $\frac{d^2y}{dx^2} - 4\frac{dy}{dx} + 4y = 0$. The auxiliary equation $m^2 - 4m + 4 = (m - 2)^2 = 0$ has a repeated root $m = 2$, so $y = (A + Bx)e^{2x}$.

Exam tips

- Learn the **hyperbolic identities** and derivatives, and use the inverse hyperbolic functions in integration.
- Solve a linear **differential equation** as complementary function + particular integral, then apply the boundary conditions.
- Use **de Moivre's theorem** for powers and roots of complex numbers and to derive trigonometric identities.
- Reduce an integral to a standard form by a stated substitution.

Further Mechanics

A-Level Further Mathematics

This handout covers Topic 3: **Further Mechanics**. It extends mechanics to projectiles, rigid bodies, circular motion, elastic strings, variable forces and collisions. Take $g = 10 \text{ m s}^{-2}$.

Motion of a projectile

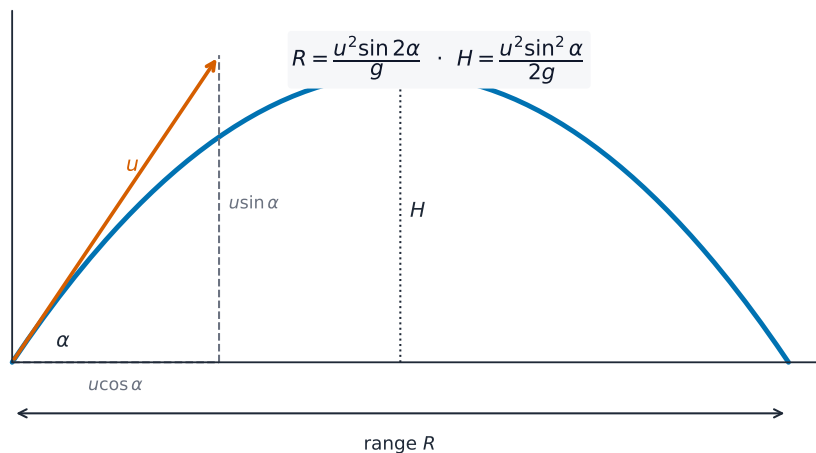


Water jets follow parabolic paths — the classic projectile motion under gravity.

A **projectile** moves freely under gravity, so it has **constant acceleration** g downwards and no horizontal acceleration. Treat the horizontal and vertical motions separately. If it is launched at speed u and angle α :

$$\text{horizontal: } x = u \cos \alpha \cdot t, \quad \text{vertical: } y = u \sin \alpha \cdot t - \frac{1}{2}gt^2.$$

Eliminating t gives the **Cartesian equation** of the **trajectory** (the path), which is a parabola. For a fixed launch speed, all the trajectories lie under a **bounding parabola** (the envelope of reachable points). The range on level ground is $\frac{u^2 \sin 2\alpha}{g}$ and the greatest height is $\frac{u^2 \sin^2 \alpha}{2g}$.



The path is a parabola; the launch speed u splits into a steady horizontal part and a vertical part slowed by gravity.

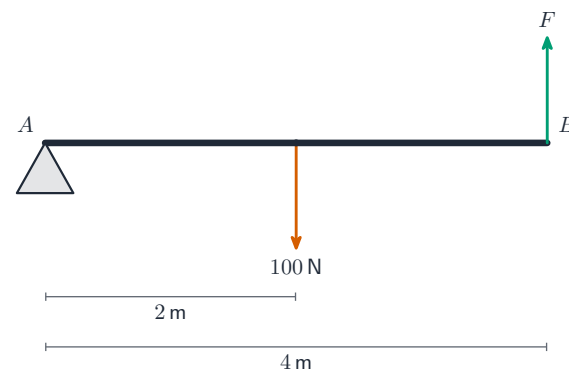
Worked example. A ball is thrown at $u = 20 \text{ m s}^{-1}$ at 30° to the horizontal. Find the range and greatest height.

$$\text{range} = \frac{20^2 \sin 60^\circ}{10} = 40 \times 0.866 = 34.6 \text{ m}, \quad \text{height} = \frac{20^2 \sin^2 30^\circ}{2 \times 10} = \frac{400 \times 0.25}{20} = 5 \text{ m}.$$

Equilibrium of a rigid body

The **moment** of a force about a point is force $\times$ perpendicular distance; it measures turning effect. The weight of a body acts at its **centre of mass**, which you can find using **symmetry** for a uniform flat shape (a **lamina**), or by treating a composite body as a set of particles.

A rigid body under **coplanar forces** is in **equilibrium** when two conditions both hold: the vector sum of the forces is zero, and the sum of the moments about any point is zero. A body may also be on the edge of **toppling** (turning over) or **sliding** (slipping).



Taking moments about the pivot A balances the turning effects: $F \times 4 = 100 \times 2$.

Worked example. A uniform beam AB of length 4 m and weight 100 N rests on a pivot at A . A vertical force F at B keeps it horizontal. Find F .

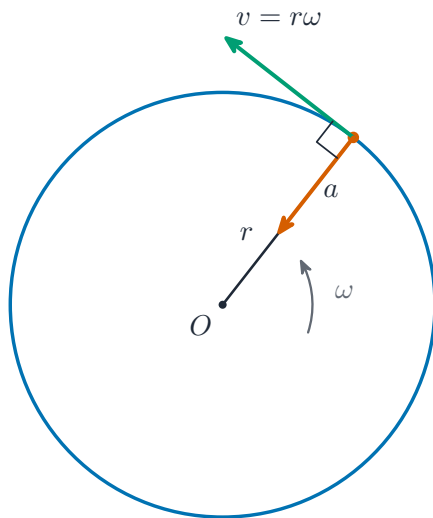
Take moments about A (the weight acts at the centre, 2 m from A):

$$F \times 4 = 100 \times 2 \Rightarrow F = 50 \text{ N}.$$

Circular motion

For a particle moving in a circle of radius r , the **angular speed** ω links to the speed by $v = r\omega$. The acceleration points towards the centre — the **centripetal acceleration** — with size

$$a = r\omega^2 = \frac{v^2}{r}.$$



The velocity points along the tangent; the acceleration points inward to the centre.



The hammer thrower is a moving diagram of this page. The taut wire pulls the ball toward the centre — that inward pull is the centripetal force — while the ball's velocity points along the tangent. Let go, and with no inward force left the ball flies straight off that tangent

In a **horizontal circle** the speed is constant — as in a **conical pendulum** (a mass swung on a string). In a **vertical circle** use energy conservation, because the speed changes with height; the **normal contact force** or

string tension provides the centripetal force.

Worked example. A particle moves in a horizontal circle of radius 2 m with angular speed 3 rad s^{-1} . Find its speed and acceleration.

$$v = r\omega = 2 \times 3 = 6 \text{ m s}^{-1}, \quad a = r\omega^2 = 2 \times 3^2 = 18 \text{ m s}^{-2}.$$

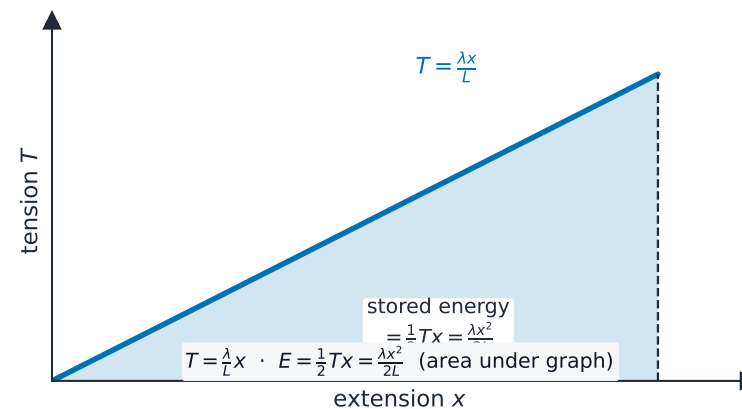
Hooke's law

Hooke's law says the tension in an elastic string or spring is proportional to its extension x :

$$T = \frac{\lambda x}{L},$$

where L is the natural length and λ is the **modulus of elasticity**. Stretching stores **elastic potential energy**:

$$E = \frac{\lambda x^2}{2L}.$$



Tension rises in proportion to extension; the shaded triangle is the stored elastic energy.

Worked example. An elastic string of natural length 2 m and modulus 50 N is stretched by 0.5 m. Find the tension and the stored energy.

$$T = \frac{50 \times 0.5}{2} = 12.5 \text{ N}, \quad E = \frac{50 \times 0.5^2}{2 \times 2} = 3.125 \text{ J}.$$

Linear motion under a variable force

When the force depends on position x , use acceleration in the form $a = v \frac{dv}{dx}$, which turns Newton's law into a **differential equation** relating v and x .

which form of acceleration?

force depends on **time** t

$$a = \frac{dv}{dt}$$

gives v as a function of t

force depends on **position** x

$$a = v \frac{dv}{dx}$$

gives v as a function of x

both come from $a = \frac{dv}{dt}$ with the chain rule; pick the one matching what the force depends on

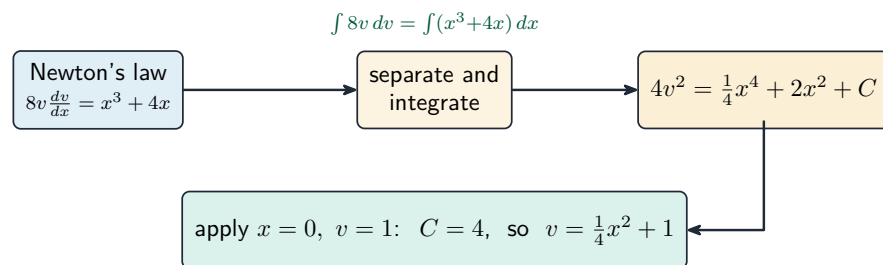
Use dv/dt when the force depends on time, $v dv/dx$ when it depends on position

Worked example. A particle of mass 8 kg moves along a line under a **variable force** of size $(x^3 + 4x)$ N acting in the direction of motion. When $x = 0$, $v = 1$. Find v in terms of x .

Newton's law gives $8v \frac{dv}{dx} = x^3 + 4x$. Separating and integrating:

$$\int 8v dv = \int (x^3 + 4x) dx \Rightarrow 4v^2 = \frac{1}{4}x^4 + 2x^2 + C.$$

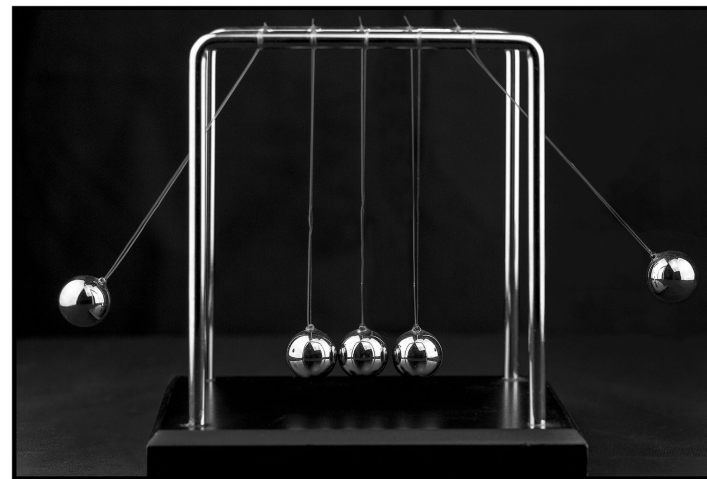
At $x = 0$, $v = 1$ gives $C = 4$, so $4v^2 = \frac{1}{4}x^4 + 2x^2 + 4 = 4\left(\frac{x^2}{4} + 1\right)^2$. Hence $v = \frac{1}{4}x^2 + 1$.



a position-dependent force becomes a **differential equation** in v and x

A position-dependent force becomes a differential equation in v and x

Momentum

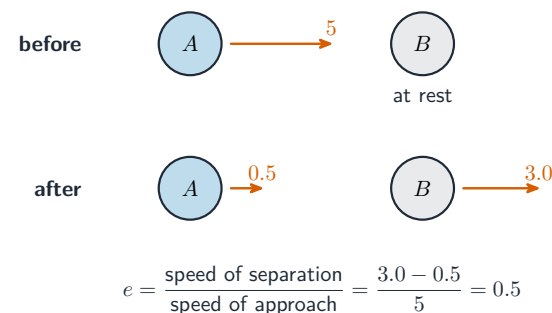


A Newton's cradle demonstrates conservation of momentum in collisions.

In a collision, total **linear momentum** is conserved — the **conservation of linear momentum**. The bounciness is measured by **Newton's experimental law**, which defines the **coefficient of restitution** e :

$$e = \frac{\text{speed of separation}}{\text{speed of approach}}, \quad 0 \leq e \leq 1.$$

Here $e = 1$ is perfectly elastic (no energy lost) and $e = 0$ is inelastic (the bodies stay together). In an **oblique impact**, resolve the velocities along and perpendicular to the line of impact, applying restitution along it.



The coefficient of restitution e compares how fast the bodies separate with how fast they approached.

Worked example. A sphere A of mass 2 kg moving at 5 m s^{-1} hits a stationary sphere B of mass 3 kg, with $e = 0.5$. Find the speeds afterwards.

Momentum: $2(5) = 2v_A + 3v_B$, so $2v_A + 3v_B = 10$. Restitution: $v_B - v_A = 0.5(5) = 2.5$. Solving together gives $v_A = 0.5 \text{ m s}^{-1}$ and $v_B = 3.0 \text{ m s}^{-1}$.

Exam tips

- For **projectiles**, resolve into horizontal (constant velocity) and vertical ($a = g$) motion, linked by the same time.
- For a **rigid body in equilibrium**, take moments about a point that removes an unknown force.
- For **circular motion**, use $F = mv^2/r = m\omega^2 r$ towards the centre; in a vertical circle check the minimum speed at the top.
- With a **variable force**, use $a = v \frac{dv}{dx}$ and integrate; elastic PE $= \frac{\lambda x^2}{2L}$.

Further Probability & Statistics

A-Level Further Mathematics

This handout covers Topic 4: **Further Probability & Statistics**.

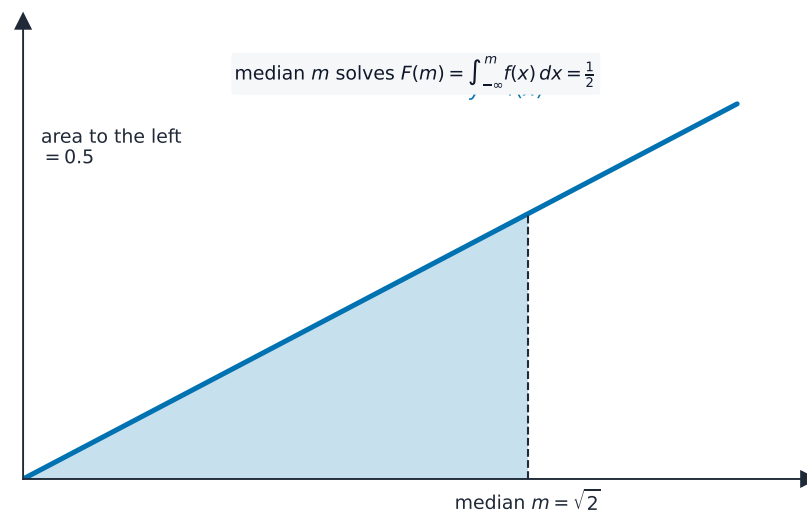
It adds continuous distributions, small-sample inference, the chi-squared and non-parametric tests, and probability generating functions.

Continuous random variables

A continuous variable X is described by a **probability density function**

$f(x)$, which may be defined piecewise. The probability over a range is the area under f , and the mean of any function of X is

$$E(g(X)) = \int g(x) f(x) dx.$$



The median sits where the area under $f(x)$ to its left is exactly 0.5.

The **cumulative distribution function**

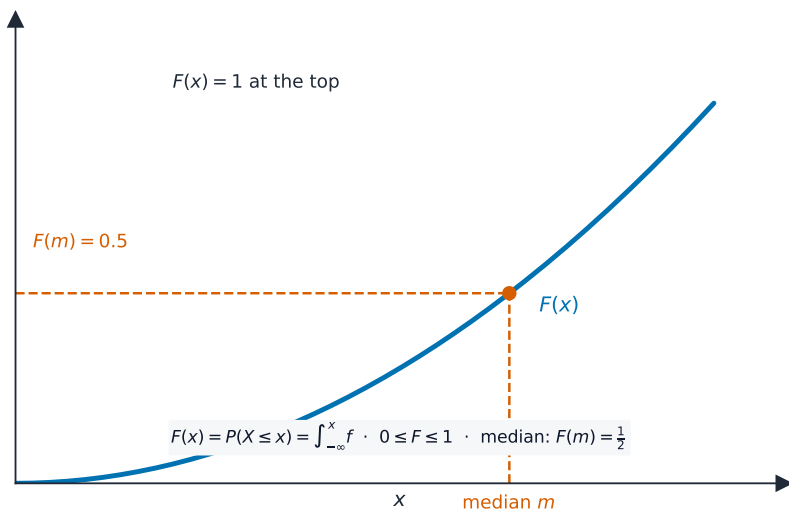
$F(x) = P(X \leq x)$ is the

running total: $F(x) = \int_{-\infty}^x f(t) dt$, and $f(x) = F'(x)$. Use F to find probabilities and

percentiles

(for example, the **median**

solves $F(x) = 0.5$).



The cumulative graph $F(x)$ rises from 0 to 1; the height 0.5 is reached at the median.

Worked example. A variable has $f(x) = \frac{1}{2}x$ for $0 \leq x \leq 2$. Find the median.

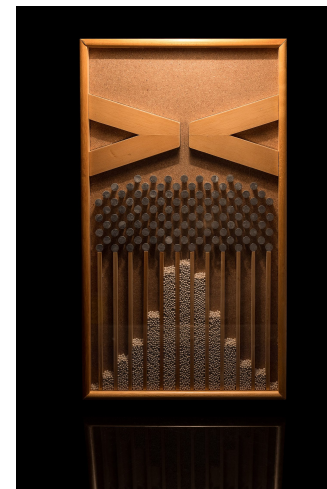
The cumulative distribution function is $F(x) = \int_0^x \frac{1}{2}t \, dt = \frac{1}{4}x^2$. Set $F(m) = 0.5$:

$$\frac{1}{4}m^2 = 0.5 \Rightarrow m^2 = 2 \Rightarrow m = \sqrt{2} = 1.41.$$

The distribution of a related variable. If $Y = g(X)$, find Y 's distribution through its cumulative function. For $Y = X^2$: $F_Y(y) = P(X^2 \leq y) = P(-\sqrt{y} \leq X \leq \sqrt{y}) = F_X(\sqrt{y}) - F_X(-\sqrt{y})$, then differentiate for $f_Y = F_Y'$. When g is monotonic increasing there is a shortcut, $F_Y(y) = F_X(g^{-1}(y))$.

Worked example. With $f_X(x) = \frac{1}{2}x$ on $[0, 2]$ (so $F_X(x) = \frac{1}{4}x^2$), let $Y = X^2$. For $0 \leq y \leq 4$, $F_Y(y) = F_X(\sqrt{y}) = \frac{1}{4}y$, so $f_Y(y) = F_Y'(y) = \frac{1}{4}$ – that is, Y is uniform on $[0, 4]$.

Inference using normal and t-distributions



A Galton board: balls falling through pins pile up into the bell-shaped normal distribution.

When a sample is small and the population variance is unknown, base your **hypothesis test** on the t -distribution instead of the normal. The same idea gives a **confidence interval** for the mean:

$$\bar{x} \pm t \frac{s}{\sqrt{n}},$$

where t comes from the t -tables with $n - 1$ degrees of freedom. To compare two populations, use a **two-sample** (2-sample) or **paired-sample** t -test, after finding a **pooled estimate** of the shared variance when appropriate.

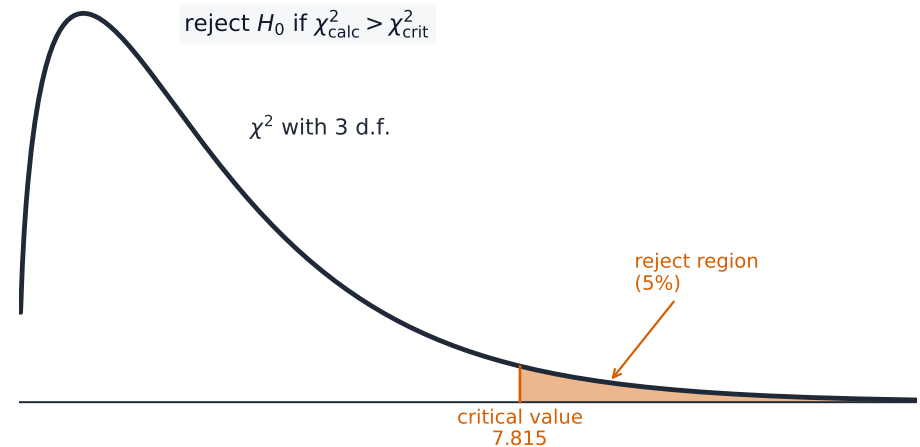
Chi-squared tests

A χ^2 -test (**chi-squared test**) compares observed counts O with expected counts E from a **theoretical distribution**:

$$\chi^2 = \sum \frac{(O - E)^2}{E}.$$

Compare this with a table value for the right number of **degrees of freedom**.

Two uses: a **goodness of fit** test (does the data follow the proposed model?), and a test for **independence** of two variables in a **contingency table**.

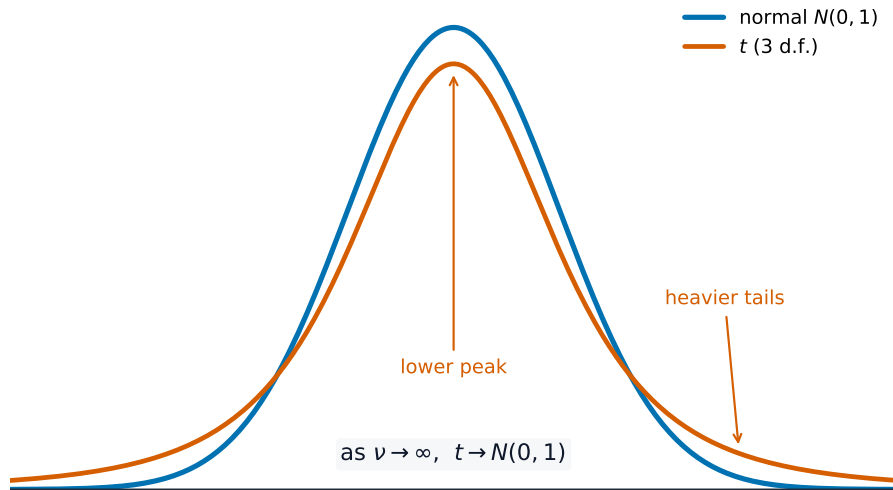


The test rejects the model when χ^2 exceeds the critical value, landing in the shaded 5% tail.

Worked example. Four equally likely categories give observed counts 20, 30, 25, 25 (so each expected count is 25). Test the fit at the 5% level.

$$\chi^2 = \frac{(20 - 25)^2 + (30 - 25)^2 + 0 + 0}{25} = \frac{25 + 25}{25} = 2.$$

With $4 - 1 = 3$ degrees of freedom the table value is 7.815. Since $2 < 7.815$, do not reject the model.



For a small sample the t -distribution is flatter with heavier tails, so its critical values are larger.

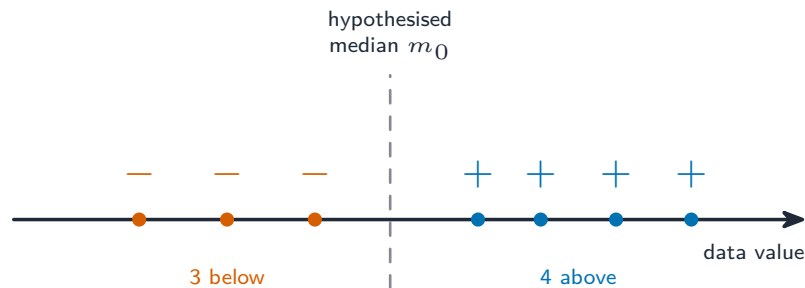
Worked example. A sample of $n = 10$ has mean $\bar{x} = 50$ and standard deviation $s = 4$. Find a 95% confidence interval for the mean (use $t = 2.262$ for 9 degrees of freedom).

$$50 \pm 2.262 \times \frac{4}{\sqrt{10}} = 50 \pm 2.86 \Rightarrow (47.1, 52.9).$$

Non-parametric tests

A **non-parametric test** makes no assumption that the data is normal, so it is useful when that assumption fails. The basic ones are:

- the **sign test** : count how many values fall above and below a proposed median, and test those counts with a binomial model;
- the **Wilcoxon signed-rank test** (the **matched-pairs test** for paired data), which also uses the sizes of the differences, not just their signs;
- the **Wilcoxon rank-sum test** , for comparing two separate samples.



The sign test counts how many values fall above and below the proposed median, then tests those counts with a binomial model

Worked example. Test whether a median is 5. In a sample of 10 values (none equal to 5), 9 lie **above** 5 and 1 lies below. Test at the 5% level (two-tailed).

Under H_0 (median = 5) the number above follows $B(10, 0.5)$. The observed result (9 above) is extreme, so find $P(X \geq 9) = \binom{10}{9}(0.5)^{10} + (0.5)^{10} = \frac{11}{1024} = 0.0107$. For a two-tailed test compare with $\frac{1}{2}(5\%) = 0.025$. Since $0.0107 < 0.025$, **reject** H_0 : there is evidence the median is not 5.

Probability generating functions



Dice: a starting point for probability and discrete random variables.

The **probability generating function** of a discrete variable X is

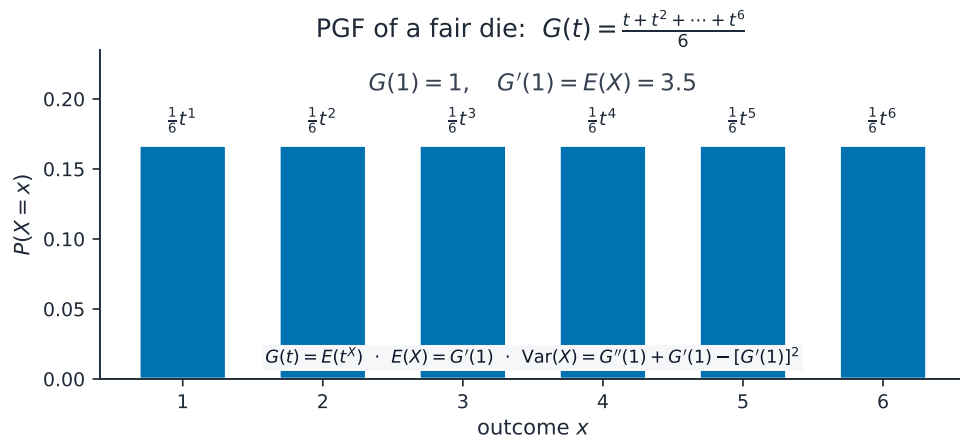
$$G(t) = E(t^X) = \sum_x P(X = x) t^x.$$

It packs the whole distribution into one function. The mean and variance come from its derivatives at $t = 1$: $E(X) = G'(1)$ and $\text{Var}(X) = G''(1) + G'(1) - (G'(1))^2$. Also, the PGF of a sum of independent variables is the product of their PGFs.

Worked example. X has $P(X = 0) = 0.5$, $P(X = 1) = 0.3$, $P(X = 2) = 0.2$. Find $E(X)$ using the PGF.

Here $G(t) = 0.5 + 0.3t + 0.2t^2$, so $G'(t) = 0.3 + 0.4t$ and

$$E(X) = G'(1) = 0.3 + 0.4 = 0.7.$$



PGF of a fair die packs equal masses at 1 through 6 into $G(t)$

Exam tips

- For a **continuous random variable**, the pdf integrates to 1 over its range, and $E(X) = \int x f(x) dx$.
- Use the **t -distribution** when the sample is small and the population variance is unknown; state the degrees of freedom.
- For a **chi-squared** test compute $\sum (O - E)^2 / E$, compare with the critical value at the right degrees of freedom, and **combine classes with $E < 5$** .
- State H_0 and H_1 and give the conclusion **in context** for every test.