

# Transformations and vectors

## IGCSE Mathematics

This handout covers Topic 7, Transformations and vectors. Parts marked (**Extended**) are only tested on the Extended papers; everything else is for both levels. Vectors as a whole topic are Extended.

## Transformations



*Geometric tiles are built by reflecting, rotating and translating shapes.*

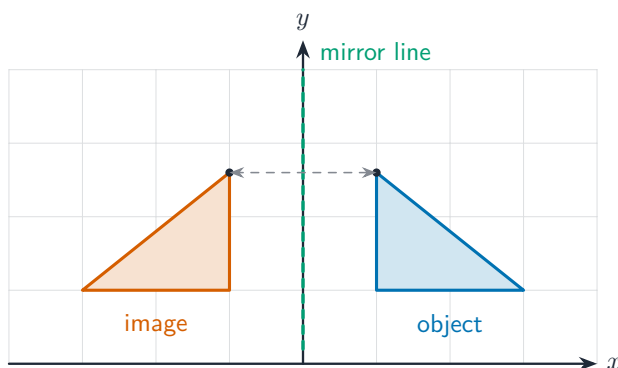
A **transformation** 变换 changes the position or size of a shape. The original is the *object* and the result is the *image*. There are four types. When asked to **describe** one, you must name the type and give all the details below.

### Reflection

A **reflection** 反射 flips the shape over a **mirror line** 对称轴. Each image point is the same distance from the line as the object point, on the other side.

- To describe it, give the **equation of the mirror line** (for Core, a horizontal or vertical line; Extended allows any line such as  $y = x$ ).

**Worked example.** Reflect the point  $(3, 2)$  in the  $y$ -axis. Only the sign of  $x$  changes: the image is  $(-3, 2)$ . (In the  $x$ -axis it would be  $(3, -2)$ .)



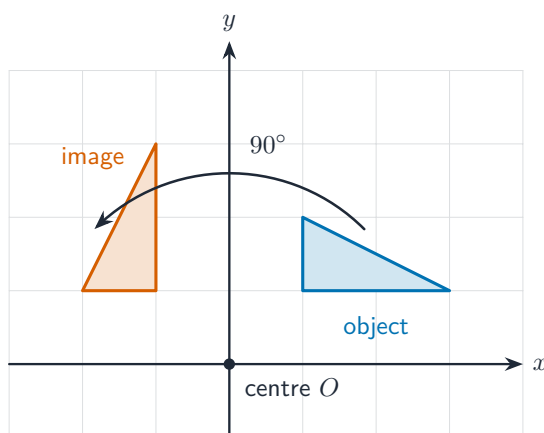
*A reflection flips the shape over a mirror line (here the  $y$ -axis); each point and its image are the same distance from the line.*

## Rotation

A **rotation** 旋转 turns the shape about a fixed point, the **centre** 中心 of rotation, through multiples of  $90^\circ$ .

- To describe it, give the centre, the angle, and the direction (clockwise or anticlockwise).

**Worked example.** Rotate  $(3, 1)$  by  $90^\circ$  anticlockwise about the origin. The rule is  $(x, y) \rightarrow (-y, x)$ , so the image is  $(-1, 3)$ .



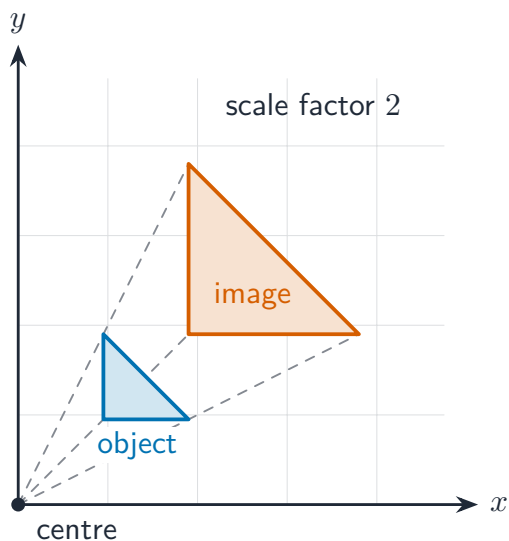
*A rotation turns the shape about a fixed centre — here  $90^\circ$  anticlockwise about the origin.*

## Enlargement

An **enlargement** 放大 changes the size by a **scale factor** 比例因子  $k$ , measured from a fixed centre. Each distance from the centre is multiplied by  $k$ .

- To describe it, give the centre and the scale factor. A **fractional** scale factor (between 0 and 1) makes the shape smaller. For Extended,  $k$  may also be **negative** (the image appears on the other side of the centre).

**Worked example.** Enlarge  $(1, 2)$  from the origin by scale factor 2. Multiply both coordinates: the image is  $(2, 4)$ .

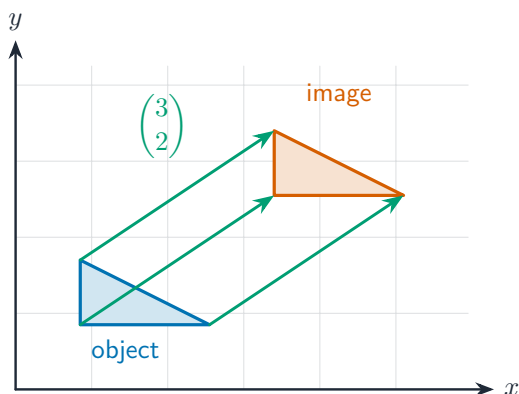


*An enlargement multiplies every distance from the centre by the scale factor (here 2).*

## Translation

A **translation** 平移 slides the shape with no turning, by a **vector** 向量 written as a **column vector** 列向量  $\begin{pmatrix} x \\ y \end{pmatrix}$  ( $x$  across,  $y$  up).

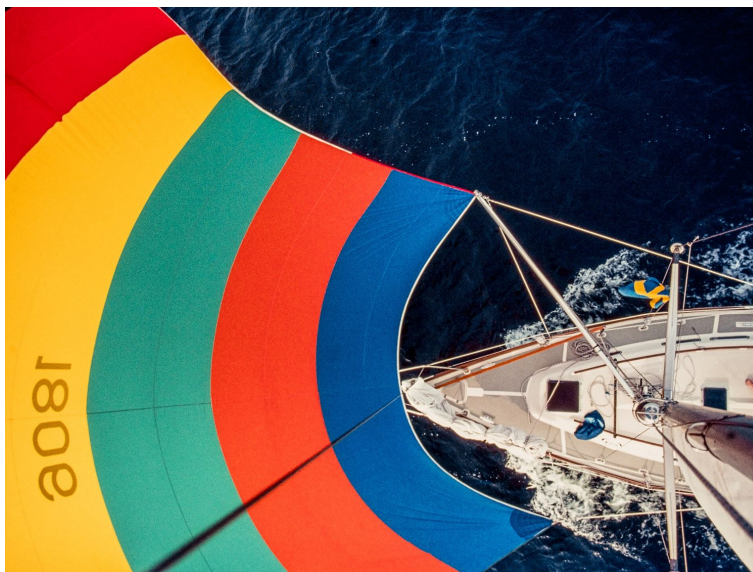
**Worked example.** Translate  $(5, 3)$  by  $\begin{pmatrix} -2 \\ 4 \end{pmatrix}$ : move 2 left and 4 up to get  $(3, 7)$ .



*A translation slides every point by the same column vector, with no turning.*

(**Extended:** a question may ask you to combine two transformations and describe the single transformation that has the same effect.)

## Vectors in two dimensions (Extended)



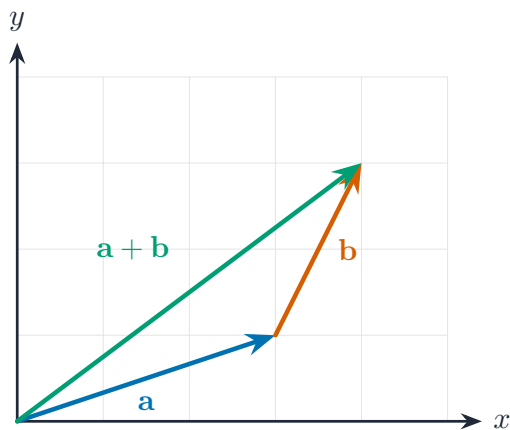
*Forces like wind are vectors, with both size and direction.*

A vector has both size and direction. It can be written as a column vector, as  $\overrightarrow{AB}$  (from  $A$  to  $B$ ), or in bold as  $\mathbf{a}$ .

- **Add or subtract** by working on the top and bottom numbers separately.
- **Multiply by a scalar** 标量 (an ordinary number) by multiplying both numbers.

**Worked example.** If  $\mathbf{a} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$  and  $\mathbf{b} = \begin{pmatrix} 2 \\ -4 \end{pmatrix}$ , then

$$\mathbf{a} + \mathbf{b} = \begin{pmatrix} 5 \\ -3 \end{pmatrix}, \quad 3\mathbf{a} = \begin{pmatrix} 9 \\ 3 \end{pmatrix}.$$



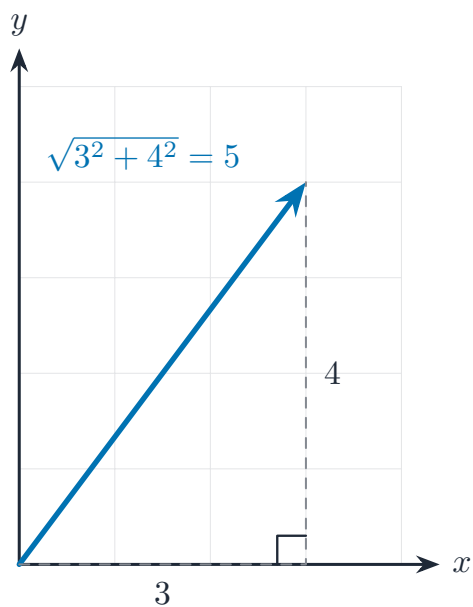
*Adding by the triangle law: draw **b** from the tip of **a**, and **a + b** runs from start to finish.*

## Magnitude of a vector (Extended)

The **magnitude** 模 (length) of a vector is found with Pythagoras. For  $\begin{pmatrix} x \\ y \end{pmatrix}$ ,

$$\left| \begin{pmatrix} x \\ y \end{pmatrix} \right| = \sqrt{x^2 + y^2}.$$

**Worked example.** The magnitude of  $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$  is  $\sqrt{3^2 + 4^2} = \sqrt{25} = 5$ .



The magnitude of  $\begin{pmatrix} 3 \\ 4 \end{pmatrix}$  comes from Pythagoras on its horizontal and vertical parts.

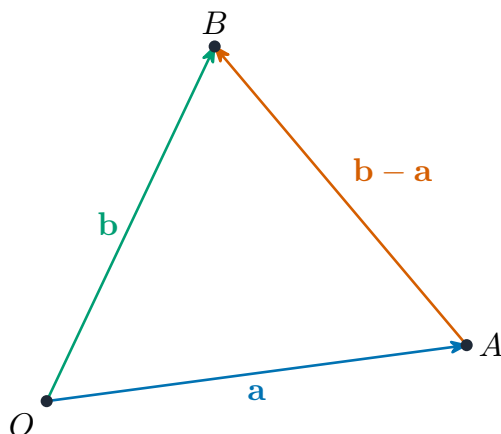
## Vector geometry (Extended)

A vector can be drawn as a **directed line segment** 有向线段 (an arrow). The **position vector** 位置向量 of a point is the vector from the origin  $O$  to that point.

A key idea: the vector from  $A$  to  $B$  is

$$\overrightarrow{AB} = \mathbf{b} - \mathbf{a},$$

where  $\mathbf{a}$  and  $\mathbf{b}$  are the position vectors of  $A$  and  $B$ .



To go from  $A$  to  $B$ , travel back along  $\mathbf{a}$  to  $O$  then forward along  $\mathbf{b}$ : so  $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$ .

Two vectors are **parallel** 平行 if one is a scalar multiple of the other (for example  $\overrightarrow{AB} = 2\overrightarrow{CD}$ ). Three points are **collinear** 共线 (in a straight line) if the vectors between them are parallel and share a point. You can express any vector in terms of two **coplanar** 共面 vectors.

**Worked example.**  $O$  is the origin, with  $\overrightarrow{OA} = \mathbf{a}$  and  $\overrightarrow{OB} = \mathbf{b}$ .  $M$  is the midpoint of  $AB$ . Find  $\overrightarrow{OM}$  in terms of  $\mathbf{a}$  and  $\mathbf{b}$ .

$$\overrightarrow{OM} = \mathbf{a} + \frac{1}{2}\overrightarrow{AB} = \mathbf{a} + \frac{1}{2}(\mathbf{b} - \mathbf{a}) = \frac{1}{2}(\mathbf{a} + \mathbf{b}).$$

## Exam tips

- Fully **describe** each transformation: a translation (a vector), a reflection (the mirror line), a rotation (centre, angle *and* direction), an enlargement (centre and scale factor).

- A **negative** scale factor turns the image upside down through the centre; a fractional one (between 0 and 1) makes it smaller.
- Add vectors tip-to-tail — add the top numbers, then the bottom numbers. The **magnitude** of a vector comes from Pythagoras on its components.
- $\overrightarrow{AB} = \mathbf{b} - \mathbf{a}$  (end minus start). Two vectors are **parallel** if one is a scalar multiple of the other.