Functions Involving Parameters, Vectors, and Matrices

AP Precalculus

Parametric Functions

A roller coaster: vectors and parametric motion describe path and velocity together

Unit 4 is part of the course but is not assessed on the AP Precalculus Exam (which covers only Units 1–3); it builds tools used in later courses.

A **parametric function** 参数函数 describes a curve by giving both coordinates as functions of a third variable, the **parameter** 参数 t: $\big(x(t), y(t)\big)$. As t runs through its domain, the point traces a path – and, unlike $y = f(x)$, the path may loop or cross itself.

Parametric Functions and Planar Motion

Reading t as **time**, a parametric function models motion in the plane. $x(t)$ and $y(t)$ give the horizontal and vertical position; the point's **direction** of travel is the order in which the curve is drawn as t increases (mark it with arrows).

Parametric Functions and Rates of Change

Over an interval of t, the average rate of change of x is $\dfrac{\Delta x}{\Delta t}$ and of y is $\dfrac{\Delta y}{\Delta t}$. Their signs tell you which way the point moves (right/left, up/down); together they describe the motion's speed and direction along the path.

Parametrically Defined Circles and Lines

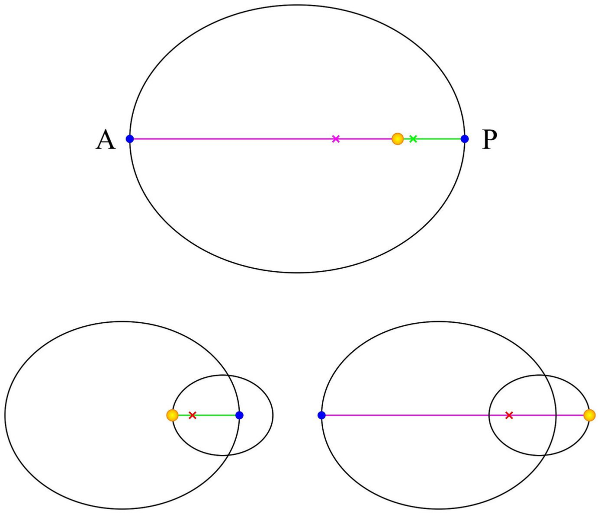
A **circle** of radius R centered at (h, k) is $x = h + R\cos t,\ y = k + R\sin t$. A **line** through (x_0, y_0) with direction (a, b) is $x = x_0 + at,\ y = y_0 + bt$. Adjusting the coefficients changes the start point, speed, and direction of tracing.

Implicitly Defined Functions

An **implicit** 隐式 equation relates x and y without solving for either, such as $x^2+y^2 = 25$. Its graph may fail the vertical-line test (not a function), so it is often split into pieces or described parametrically.

Conic Sections

Conic sections 圆锥曲线 – circles, ellipses, parabolas, and hyperbolas – are the curves from slicing a cone, each given by a quadratic equation in x and y. Their standard forms reveal centers, vertices, axes, and asymptotes.



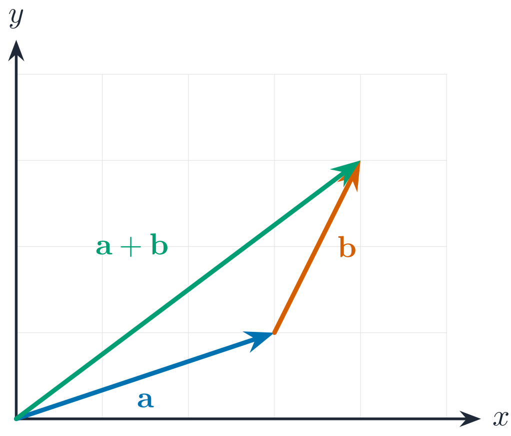
Elliptical orbits: conic sections describe closed paths with two foci — a key parametric model

Parametrizing Implicit Curves

Many implicit curves can be **parametrized** – rewritten as $\big(x(t), y(t)\big)$ – which makes them easier to graph and to treat as motion. The circle above is the basic example; ellipses use $x = h + A\cos t,\ \ y = k + B\sin t$.

Vectors

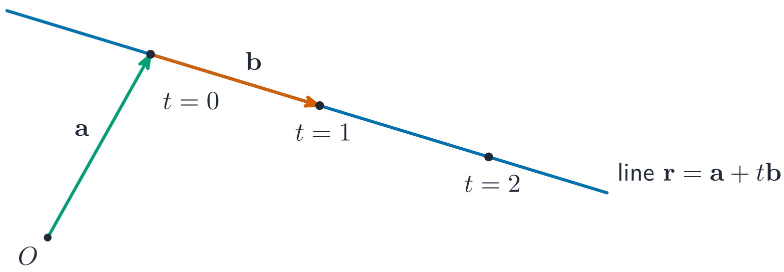
A **vector** 向量 has both **magnitude** 大小 (length) and **direction**. Write it by components $\langle a, b\rangle$. Add vectors component-by-component; scale by multiplying each component; the magnitude is $\sqrt{a^2+b^2}$. Vectors model displacements, velocities, and forces.



Adding vectors by the triangle law

Vector-Valued Functions

A **vector-valued function** 向量值函数 outputs a vector for each input, e.g. $\vec{r}(t) = \langle x(t), y(t)\rangle$ – the same information as a parametric function, packaged as a moving position vector.



A vector-valued line: start at a, then slide by t lots of the direction b

Matrices

A **matrix** 矩阵 is a rectangular array of numbers. Add matrices of the same size entry-by-entry; multiply a matrix by a compatible one by combining rows with columns. Matrices store and transform data compactly.

The Inverse and Determinant of a Matrix

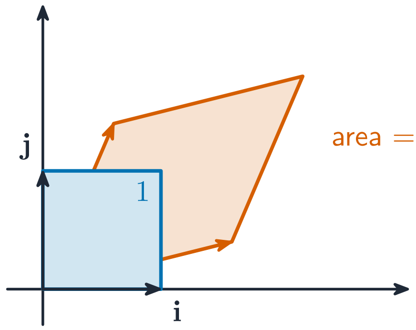
The **determinant** 行列式 of a 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $ad-bc$. A matrix has an **inverse** 逆矩阵 exactly when its determinant is nonzero; the inverse undoes the matrix, and it solves matrix equations (like a reciprocal for numbers).

Worked example. For $A = \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}$, the determinant is $ad-bc = (3)(4)-(1)(2) = 10$. Since it is nonzero, A is invertible, and $A^{-1} = \dfrac{1}{10}\begin{bmatrix} 4 & -1 \\ -2 & 3 \end{bmatrix}$ (swap the diagonal, negate the off-diagonal, divide by the determinant).

Linear Transformations and Matrices

A 2×2 matrix acts as a **linear transformation** 线性变换 of the plane – rotating, reflecting, stretching, or shearing points – by multiplying each position vector. The determinant measures how the transformation scales area (and its sign tells whether orientation flips).

Worked example. The matrix $\begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$ stretches the plane by 2 horizontally and 3 vertically, sending the vector $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ to $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$. Its determinant $2 \times 3 = 6$ means every area is multiplied by 6, so the unit square becomes a 2×3 rectangle.



A 2x2 matrix maps the unit square to a parallelogram; the determinant is the area scale

Matrices as Functions

Because a matrix maps input vectors to output vectors, it **is** a function on the plane. Composing transformations corresponds to multiplying their matrices, and the inverse matrix reverses the mapping.

Matrices Modeling Contexts

Matrices model systems that step forward in stages – for example, populations moving between states each year. Repeated multiplication by a **transition matrix** advances the model one step at a time, so matrix powers predict long-run behavior.

Exam tips

- A **parametric** function gives $x(t)$ and $y(t)$ separately — track the direction of motion as t increases.
- A **vector** has magnitude and direction; add component-by-component and find magnitude with $\sqrt{a^2 + b^2}$.
- The **determinant** of a 2×2 matrix is $ad - bc$ (mind the minus sign); it scales area under the transformation.
- A 2×2 matrix transforms the plane (rotate, reflect, stretch); an inverse exists only when the determinant is non-zero.
- Note this unit is not tested on the AP Precalculus Exam (Units 1–3 only), but it underpins later courses.