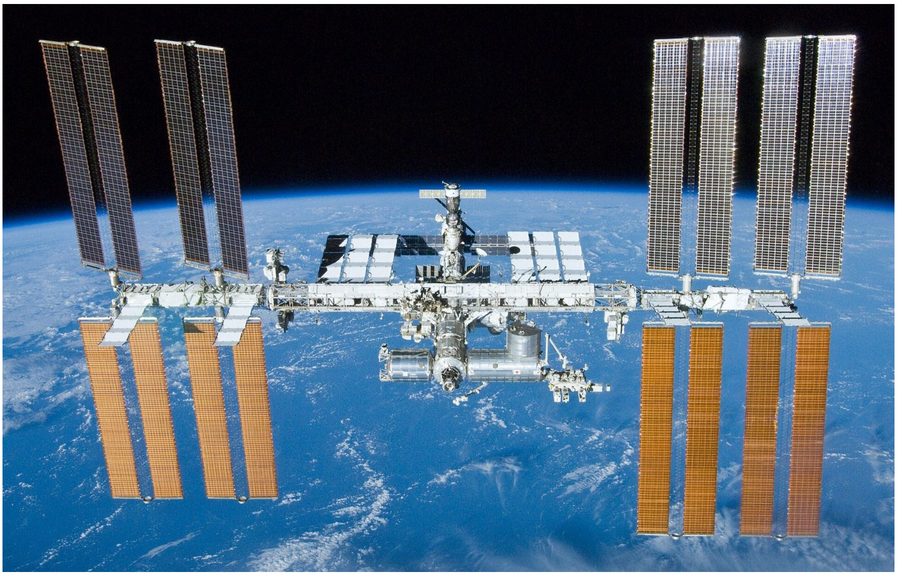
Trigonometric and Polar Functions

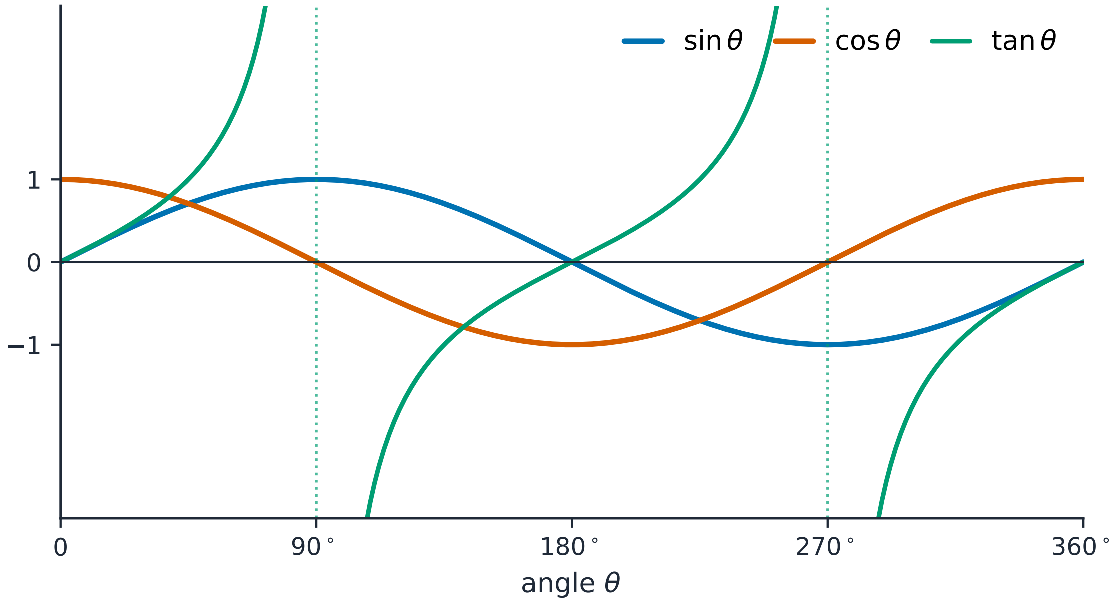
AP Precalculus

Periodic Phenomena



The ISS in orbit: trigonometric and polar functions describe periodic and circular motion

A relationship is **periodic** 周期性 if its output pattern repeats at regular input steps. The **period** 周期 is the smallest positive k with $f(x+k)=f(x)$ for all x. You can build the whole graph by copying a single **cycle** 周期段, and estimate the period by finding how far apart the pattern repeats. Within each cycle a periodic function still has intervals of increase/decrease and maxima/minima.



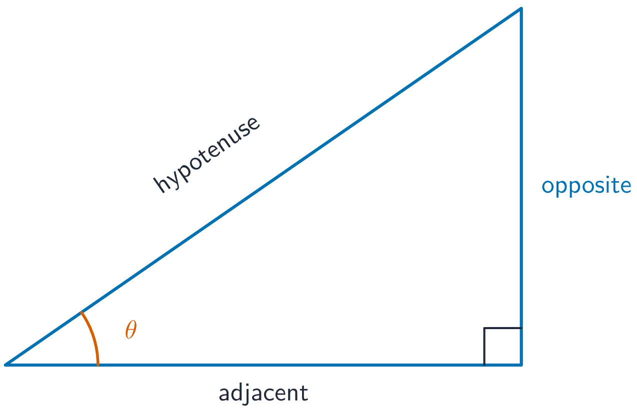
sin and cos wave between -1 and 1; tan breaks at 90 and 270 degrees

As a Ferris wheel turns steadily, your height rises and falls over and over — a real sine wave in time

Sine, Cosine, and Tangent

An angle is in **standard position** 标准位置 when its vertex is at the origin and its initial ray lies on the positive x-axis. Its **radian** 弧度 measure is the arc length on a unit circle. For the point P where the terminal ray meets a circle of radius r:

$$\cos\theta = \frac{x}{r}, \qquad \sin\theta = \frac{y}{r}, \qquad \tan\theta = \frac{y}{x} \text{ (the slope of the terminal ray).}$$



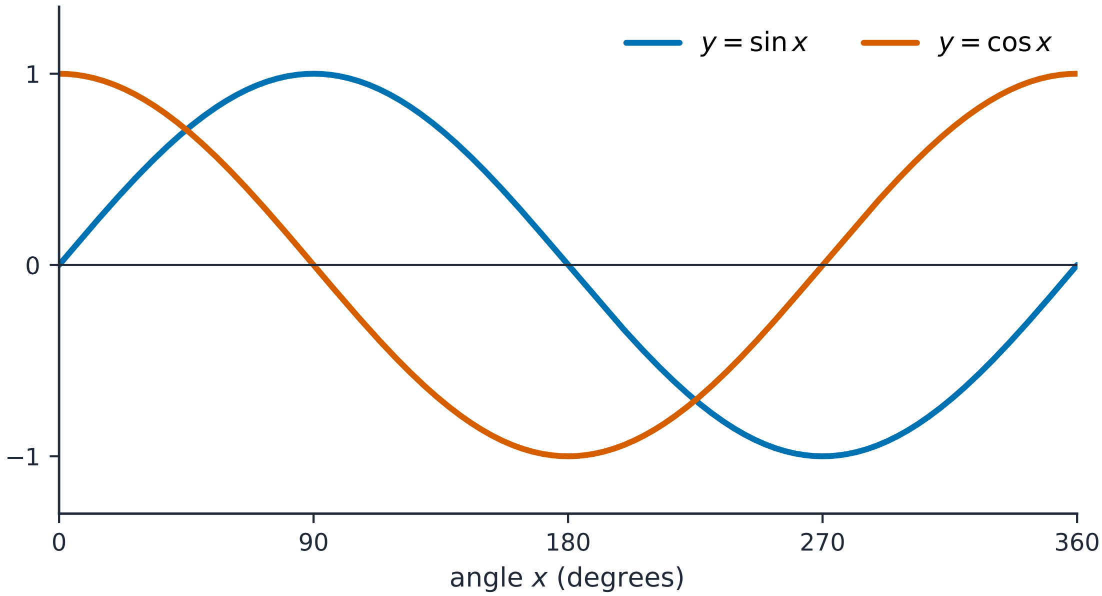
Naming the sides of a right triangle from the angle

Sine and Cosine Function Values

On a circle of radius r, $x = r\cos\theta$ and $y = r\sin\theta$. Exact values at the special angles $(0, \frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}, \ldots)$ come from **isosceles right** and **equilateral triangle** 等边三角形 geometry, with signs set by the **quadrant** 象限 of the angle.

Sine and Cosine Function Graphs

On the **unit circle** 单位圆 ($r = 1$), $\sin\theta$ is the y-coordinate and $\cos\theta$ is the x-coordinate. As θ increases, both **oscillate** 振荡 smoothly between -1 and 1 with period 2π. Sine starts at 0 (rising); cosine starts at 1. They are the same wave shifted by $\frac{\pi}{2}$.



y = sin x and y = cos x are smooth waves between -1 and 1

Sinusoidal Functions

A **sinusoidal function** 正弦型函数 is any transformation of sine (or cosine). Its key features:

- **Period** and **frequency** 频率 are reciprocals; $\sin\theta$ has period 2π.
- **Amplitude** 振幅 $=$ half the distance between the maximum and minimum output.
- **Midline** 中线 $y = d =$ the average of the maximum and minimum (the horizontal center line).

The graph alternates concave up and concave down each half-cycle.

Worked example. For $f(\theta) = 3\sin(2\theta) + 1$: the amplitude is 3, the period is $\dfrac{2\pi}{2} = \pi$, and the midline is $y = 1$. So the maximum output is $1 + 3 = 4$ and the minimum is $1 - 3 = -2$.

Sinusoidal Function Transformations

The general sinusoid is

$$f(\theta) = a\sin\big(b(\theta + c)\big) + d \quad \text{or} \quad a\cos\big(b(\theta + c)\big) + d,$$

where $|a|$ is the **amplitude**, $\dfrac{2\pi}{|b|}$ is the **period**, $-c$ is the horizontal **phase shift** 相移 (a $+c$ inside the bracket shifts the graph **left** by c), and d is the **vertical shift** (midline). A negative a reflects the wave.

Sinusoidal Function Context and Data Modeling

To model periodic data (tides, daylight, temperature): read the **period** from where the pattern repeats, get the **amplitude** and **midline** from the max and min, and set the phase shift so a peak lands at the right input. Technology fits a sinusoidal regression. Such a model is only trustworthy over its **contextual domain**.

The Tangent Function

$\tan\theta = \dfrac{\sin\theta}{\cos\theta}$ is the **slope** of the terminal ray. Its slope values repeat every half-revolution, so its period is π. It has **vertical asymptotes** where $\cos\theta = 0$ (at $\theta = \frac{\pi}{2} + k\pi$), and it is always increasing between consecutive asymptotes, switching concavity at each zero.

Inverse Trigonometric Functions

The **inverse trigonometric functions** 反三角函数 – **arcsine** 反正弦, **arccosine**, **arctangent** – reverse the trig functions, so they take a ratio and return an angle. Because sine, cosine, and tangent repeat, their domains must be **restricted** (e.g. sine to $[-\frac{\pi}{2}, \frac{\pi}{2}]$) before an inverse can exist.

Trigonometric Equations and Inequalities

Use inverse trig functions to solve trig equations. Because the functions are **periodic**, there are usually **infinitely many** solutions – write the general solution by adding multiples of the period, then keep those in the required (often contextual) domain.

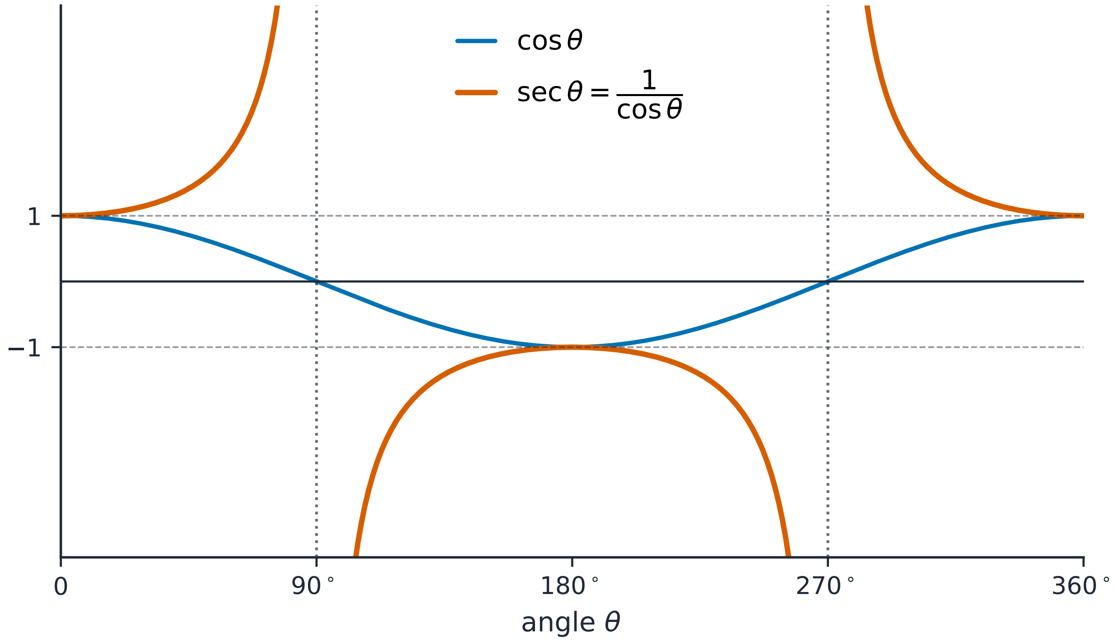
Worked example. Solve $2\sin\theta = 1$ on $[0, 2\pi)$. Then $\sin\theta = \frac{1}{2}$, and on the unit circle sine is positive in the first and second quadrants, so $\theta = \dfrac{\pi}{6}$ or $\dfrac{5\pi}{6}$. Over all reals you would add $2\pi k$ to each.

The Secant, Cosecant, and Cotangent Functions

These are the **reciprocals** 倒数 of the main three:

$$\sec\theta = \frac{1}{\cos\theta}, \qquad \csc\theta = \frac{1}{\sin\theta}, \qquad \cot\theta = \frac{1}{\tan\theta}.$$

Secant and cosecant have vertical asymptotes where cosine and sine are zero; cotangent has them where tangent is zero (where sine is zero).



sec is the reciprocal of cos, with asymptotes where cos is zero

Equivalent Representations of Trigonometric Functions

The **Pythagorean identity** 毕达哥拉斯恒等式 comes from $x^2 + y^2 = 1$ on the unit circle:

$$\sin^2\theta + \cos^2\theta = 1,$$

which rearranges into forms with sec and tan. The **sum identities** 和角公式:

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta, \qquad \cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta,$$

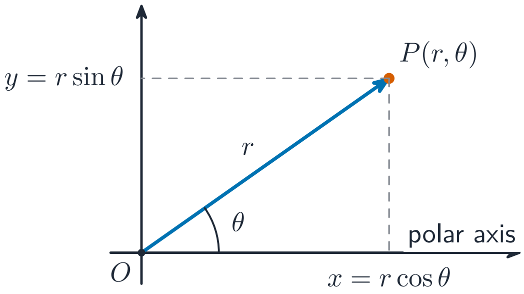
also give the difference and **double-angle** 二倍角 identities (set $\beta = \alpha$).

Trigonometry and Polar Coordinates

Polar coordinates 极坐标 locate a point by (r, θ) – distance r from the origin at angle θ. Convert with

$$x = r\cos\theta, \quad y = r\sin\theta, \qquad r = \sqrt{x^2 + y^2}, \quad \tan\theta = \frac{y}{x}.$$

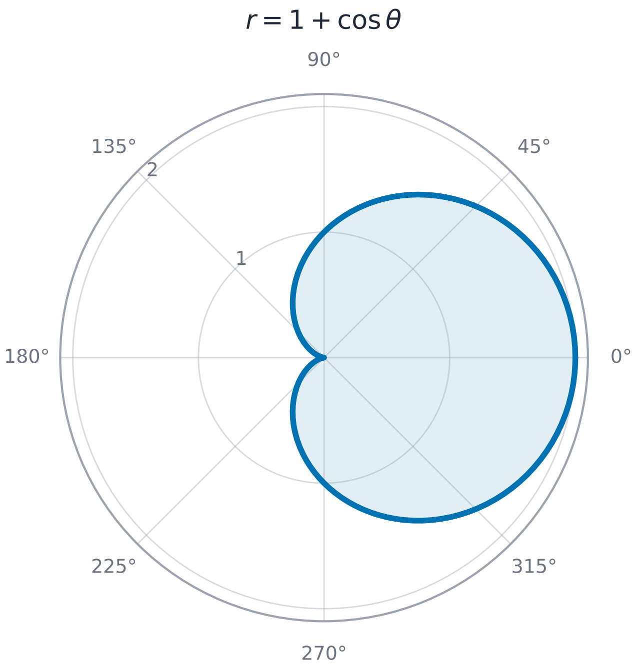
A **complex number** 复数 is the point $(x, y) = x + yi$ and can likewise be written with r and θ.



Polar coordinates give a point by its distance r and angle theta

Polar Function Graphs

The graph of $r = f(\theta)$ is all points (r, θ) as θ sweeps around. Restricting the domain of θ draws only part of the curve. As θ increases, r grows or shrinks, tracing spirals, circles, roses, and limaçons.



A cardioid r = 1 + cos theta, sketched directly in polar form

Rates of Change in Polar Functions

As θ increases, the **distance from the origin** r changes:

- r positive and increasing (or negative and decreasing) $\Rightarrow$ moving **away** from the origin;
- r positive and decreasing (or negative and increasing) $\Rightarrow$ moving **toward** the origin.

Where r switches between increasing and decreasing, the distance reaches a relative extreme. The **average rate of change** of r with respect to θ, $\dfrac{\Delta r}{\Delta \theta}$, estimates how fast the curve moves in or out over an interval.

Exam tips

- Use radians and the **unit circle**: $\cos\theta$ and $\sin\theta$ are the coordinates, always between -1 and 1.
- For a sinusoid $a\sin(b(\theta+c))+d$: $|a|$ is the amplitude, $\frac{2\pi}{|b|}$ the period, d the midline, and the phase shift is $-c$ (a $+c$ shifts left).
- Model periodic data by reading period, amplitude, and midline from the max and min.
- **Inverse trig** functions need a restricted domain and return an angle; trig equations have infinitely many solutions (add the period).
- Convert polar↔rectangular with $x = r\cos\theta$, $y = r\sin\theta$.