

Torque and Rotational Dynamics

AP Physics C: Mechanics

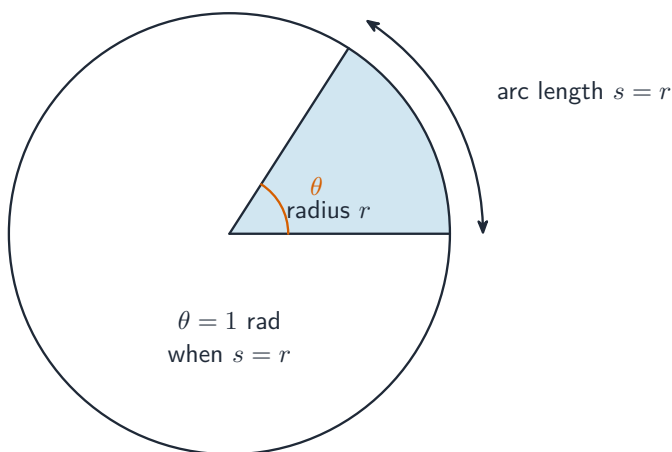
Rotational Kinematics

Rotation mirrors linear motion with angular quantities, defined as derivatives:

$$\omega = \frac{d\theta}{dt}, \quad \alpha = \frac{d\omega}{dt} = \frac{d^2\theta}{dt^2}.$$

Here θ is **angular displacement** 角位移 in **radians** 弧度, ω the **angular velocity** 角速度, and α the **angular acceleration** 角加速度. (AP works with their magnitudes and signs; the 3D vector directions are not assessed.) For constant α , the kinematic equations are the linear ones re-lettered:

$$\omega = \omega_0 + \alpha t, \quad \Delta\theta = \omega_0 t + \frac{1}{2}\alpha t^2, \quad \omega^2 = \omega_0^2 + 2\alpha \Delta\theta.$$



One radian is the angle whose arc length equals the radius

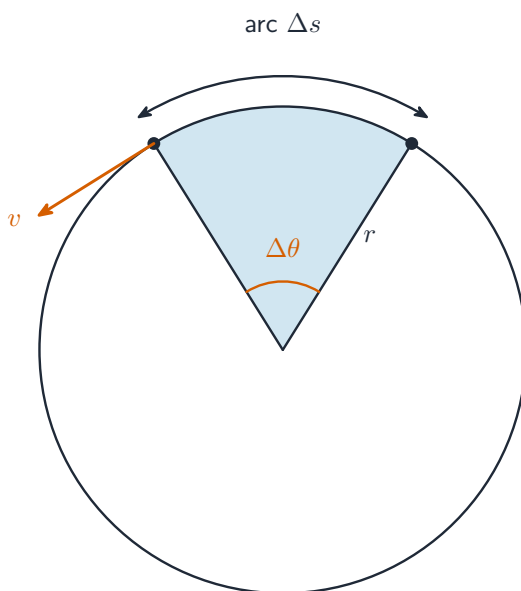
Worked example. A fan blade slows from 30 rad/s to rest with $\alpha = -6.0 \text{ rad/s}^2$: it takes $t = 5.0 \text{ s}$ and turns through $\Delta\theta = \frac{0 - 30^2}{2(-6.0)} = 75 \text{ rad}$ – about 12 revolutions.

Connecting Linear and Rotational Motion

A point at radius r from the axis moves along its arc with

$$s = r\theta, \quad v = r\omega, \quad a_t = r\alpha,$$

so points farther out move faster. A rotating point generally has *two* acceleration components at once: the **tangential acceleration** 切向加速度 $a_t = r\alpha$ (speeding up along the arc) and the **centripetal acceleration** 向心加速度 $a_c = \frac{v^2}{r} = r\omega^2$ (turning, towards the axis). They are perpendicular, so $a = \sqrt{a_t^2 + a_c^2}$.



As the radius turns through an angle, a point moves along an arc at speed v

Torque

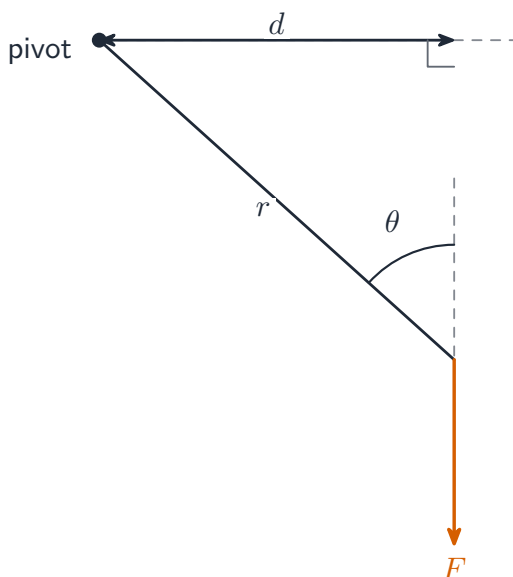
Torque 力矩 is the turning effect of a force – the rotational cause, as force is the translational cause. It is a **vector product** 矢量积:

$$\vec{\tau} = \vec{r} \times \vec{F}, \quad \tau = rF \sin \theta = Fr_{\perp},$$

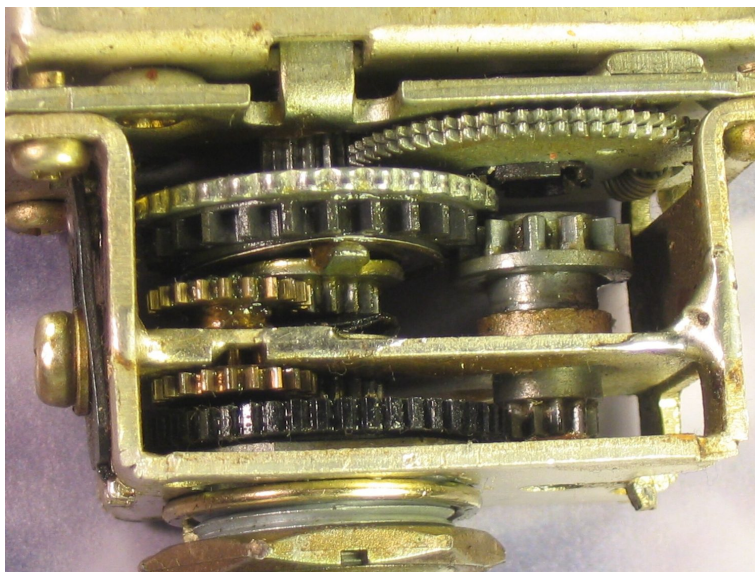
where $r_{\perp}$, the **moment arm** 力臂, is the perpendicular distance from the axis to the force's line of action. More force, applied farther from the axis, more perpendicular:

more torque. A force pointing straight at (or away from) the axis has zero torque. In a plane, give counterclockwise and clockwise torques opposite signs and add.

As a vector product, $\vec{\tau} = \vec{r} \times \vec{F}$ points **perpendicular to the plane** of $\vec{r}$ and $\vec{F}$, with its direction fixed by the **right-hand rule** 右手定则: curl your right fingers from $\vec{r}$ toward $\vec{F}$ and your thumb points along $\vec{\tau}$ – out of the page for a counterclockwise turn, into it for a clockwise one. The same rule gives the direction of angular momentum $\vec{L} = \vec{r} \times \vec{p}$.



The torque of a force depends on the perpendicular distance from the axis



A gear train: torques and rotational inertia couple through the gear ratio

Rotational Inertia

Rotational inertia 转动惯量 I measures resistance to angular acceleration – the rotational analog of mass, but it depends on *where* the mass sits. For point masses, $I = \sum m_i r_i^2$; for a continuous body,

$$I = \int r^2 dm.$$

Mass far from the axis dominates, because of the r^2 .

Worked example (rod). A uniform rod (mass M , length L) about its centre: with **linear density** 线密度 $\lambda = M/L$, $dm = \lambda dx$, so $I = \int_{-L/2}^{L/2} x^2 \frac{M}{L} dx = \frac{1}{12}ML^2$. AP also expects *nonuniform* rods: if $\lambda(x) = cx$ on $[0, L]$, first find $M = \int_0^L cx dx = \frac{1}{2}cL^2$, then $I = \int_0^L x^2(cx) dx = \frac{1}{4}cL^4 = \frac{1}{2}ML^2$ about the light end.

Worked example (disk from rings). A solid disk is a nest of rings: a ring at radius r of width dr has $dm = \frac{M}{\pi R^2} 2\pi r dr$, so

$$I = \int_0^R r^2 dm = \frac{2M}{R^2} \int_0^R r^3 dr = \frac{1}{2}MR^2.$$

The same coaxial-shell method handles cylindrical shells and annular rings.

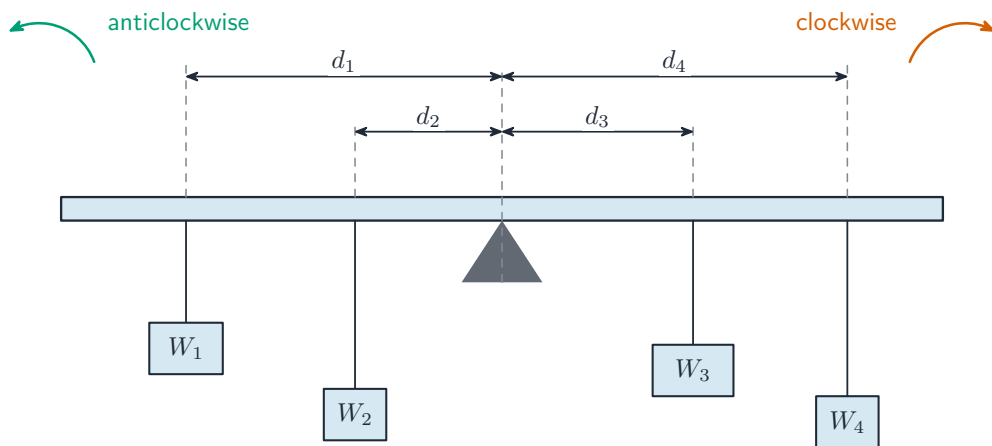
The **parallel-axis theorem** 平行轴定理 shifts any known I to a parallel axis a distance d away:

$$I' = I_{\text{cm}} + Md^2.$$

For the rod about one end: $I = \frac{1}{12}ML^2 + M\left(\frac{L}{2}\right)^2 = \frac{1}{3}ML^2$.

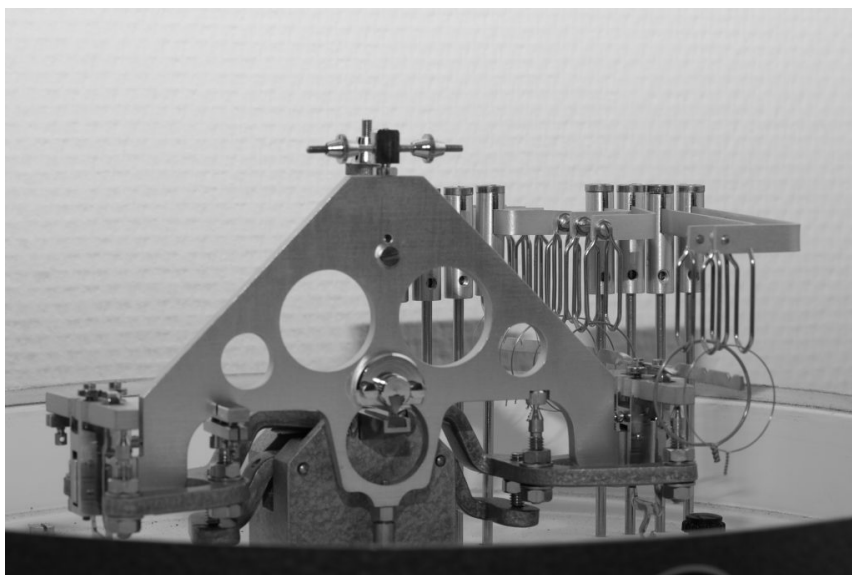
Rotational Equilibrium

Newton's first law has a rotational form: with zero net torque, angular velocity is constant – an object in **rotational equilibrium** 转动平衡 either does not rotate or rotates steadily. Full **static equilibrium** 静平衡 needs both $\sum F = 0$ and $\sum \tau = 0$ (about *any* axis). The standard move for beams and ladders: put the axis at the point where an unknown force acts, so that force drops out of the torque equation.



At balance the clockwise and anticlockwise torques about the axis are equal

Worked example. A uniform 20 kg beam, 4.0 m long, is pivoted at its left end and held horizontal by a vertical rope at its right end. A 40 kg child sits 1.0 m from the pivot. Torques about the pivot: $T(4.0) = 20g(2.0) + 40g(1.0)$, so $T = \frac{392 + 392}{4.0} = 196$ N. Then vertical force balance gives the pivot's upward push: $F_p = (60)(9.8) - 196 = 392$ N. Choosing the pivot as the axis removed F_p from the torque equation entirely.



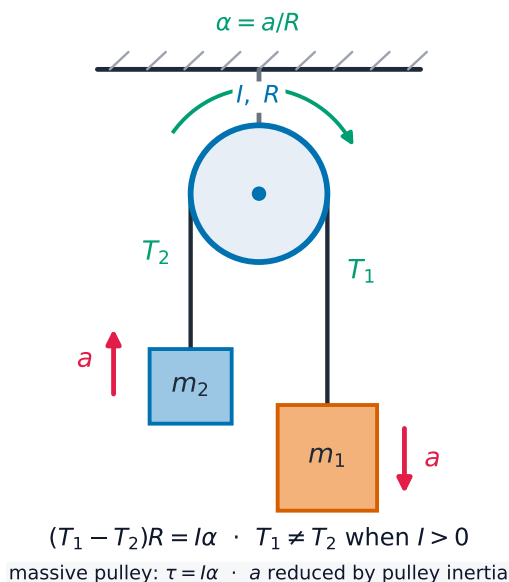
A beam balance: rotational equilibrium when clockwise and anticlockwise torques match

Newton's Second Law in Rotational Form

A net torque produces angular acceleration in proportion to the rotational inertia:

$$\alpha = \frac{\sum \tau}{I} \quad \left(\text{more generally } \sum \vec{\tau} = \frac{d\vec{L}}{dt} \right).$$

Solve rotational dynamics exactly like translational dynamics, with $\tau \leftrightarrow F$, $I \leftrightarrow m$, $\alpha \leftrightarrow a$ – free-body diagram, equations, solve.



A massive pulley: the two tensions differ because the pulley needs net torque

Worked example (massive pulley). Masses $m_1 = 4.0$ kg and $m_2 = 2.0$ kg hang from a rope over a **pulley** 滑轮 of rotational inertia $I = 0.50$ kg $\cdot$ m² and radius $R = 0.20$ m. Three Newton's-law equations, one per body:

$$m_1 g - T_1 = m_1 a, \quad T_2 - m_2 g = m_2 a, \quad (T_1 - T_2)R = I\alpha = \frac{Ia}{R}.$$

Adding them (with $I/R^2 = 12.5$ kg): $a = \frac{(m_1 - m_2)g}{m_1 + m_2 + I/R^2} = \frac{19.6}{18.5} = 1.1$ m/s². Then $T_1 = m_1(g - a) = 35$ N and $T_2 = m_2(g + a) = 22$ N. The tensions *must* differ – otherwise nothing would spin the pulley. (If a problem says the pulley is light, then $I \approx 0$ and the tensions become equal again.)

Exam skill. Rotational FRQs score the setup: separate free-body diagrams for each mass *and* the pulley, the string constraint $a = R\alpha$ stated, and the sign convention

consistent. The single most common error is assuming one tension throughout a rope that passes over a *massive* pulley.

Exam tips

- Use the rotational analogues: $\theta, \omega = \frac{d\theta}{dt}, \alpha = \frac{d\omega}{dt}$, with the constant- α equations mirroring linear kinematics.
- Convert between linear and angular with $v = r\omega$ and $a_t = r\alpha$ (plus centripetal $a_c = \frac{v^2}{r} = \omega^2 r$).
- Keep radians throughout and fix a positive sense of rotation.
- Relate torque to angular acceleration through $\tau = I\alpha$.
- Draw the rotation axis — the moment arm is the perpendicular distance to it.