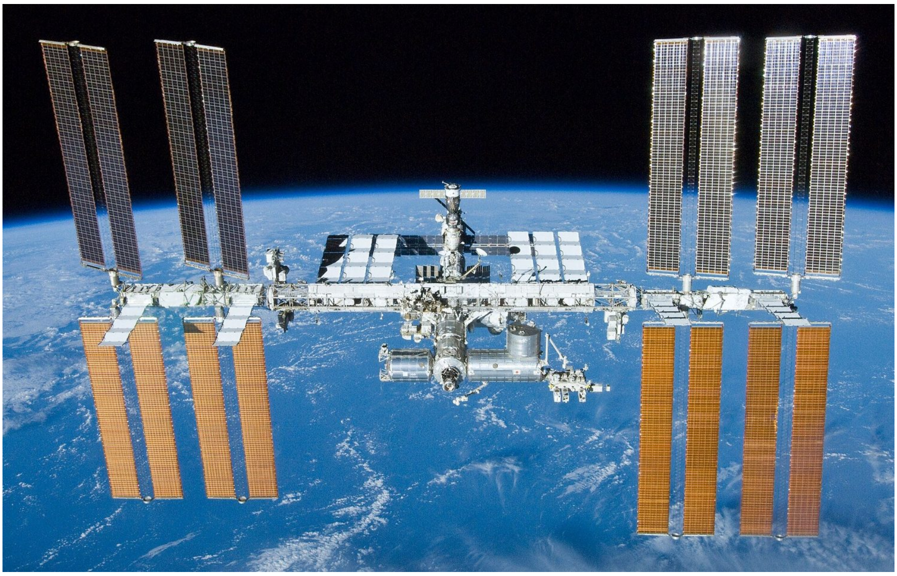
Parametric Equations, Polar Coordinates, and Vector-Valued Functions

AP Calculus BC

Defining and Differentiating Parametric Equations



The ISS in orbit: parametric equations describe position as a function of time

Parametric equations 参数方程 give x and y each as functions of a **parameter** t (often time): $x = x(t)$, $y = y(t)$. They trace a curve that need not be a function of x. The slope of the curve is found with the chain rule:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} \qquad (dx/dt \neq 0).$$

Worked example. For $x = t^2$, $y = t^3 - t$, find the slope at $t = 2$. Here $\dfrac{dx}{dt} = 2t$ and $\dfrac{dy}{dt} = 3t^2 - 1$, so $\dfrac{dy}{dx} = \dfrac{3t^2 - 1}{2t}$; at $t = 2$ this is $\dfrac{11}{4}$.

Second Derivatives of Parametric Equations

The second derivative is **not** $\dfrac{d^2y/dt^2}{d^2x/dt^2}$. Instead, differentiate the first derivative with respect to t, then divide by dx/dt again:

$$\frac{d^2y}{dx^2} = \frac{\dfrac{d}{dt}\left(\dfrac{dy}{dx}\right)}{dx/dt}.$$

Use it to test concavity of a parametric curve.

Finding Arc Lengths of Curves Given by Parametric Equations

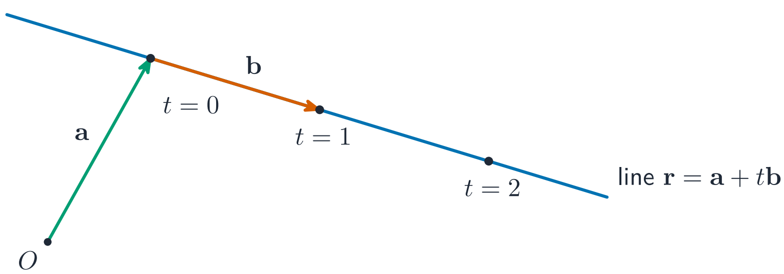
The length of a parametric curve for t from a to b is

$$L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt.$$

This is the parametric version of arc length – summing tiny hypotenuses $\sqrt{dx^2 + dy^2}$ over the parameter.

Defining and Differentiating Vector-Valued Functions

A **vector-valued function** 向量值函数 $\vec{r}(t) = \langle x(t), y(t)\rangle$ gives a position vector for each t. Differentiate **component-wise**: the **velocity** is $\vec{v}(t) = \langle x'(t), y'(t)\rangle$ and the **acceleration** is $\vec{a}(t) = \langle x''(t), y''(t)\rangle$. The **speed** is the magnitude $|\vec{v}| = \sqrt{x'^2 + y'^2}$.



A vector-valued line: start at a, then slide by t lots of the direction b

Integrating Vector-Valued Functions

Integrate a vector function **component by component**. Given acceleration or velocity plus an initial condition, integrate each component and use the condition to find the constants – recovering velocity from acceleration, or position from velocity.

Solving Motion Problems Using Parametric and Vector-Valued Functions

For a particle moving in a plane: position is $\langle x(t), y(t)\rangle$, velocity and acceleration are its derivatives, speed is $|\vec{v}|$, and the **distance traveled** over $[a, b]$ is

$$\int_a^b \sqrt{x'(t)^2 + y'(t)^2}\, dt.$$

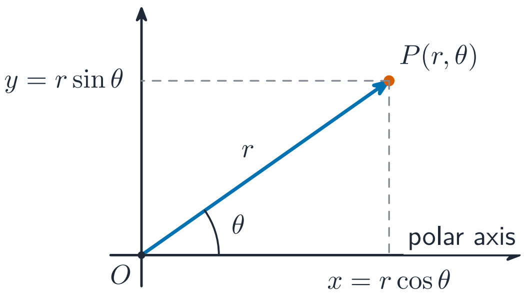
Exam skill: these plane-motion problems appear on the BC free-response nearly every year – be fluent finding speed, the position at a later time (initial point plus the integral of velocity), and total distance.

Worked example. A particle has position $\langle t^2,\ t^3 - t\rangle$. Its velocity is $\langle 2t,\ 3t^2 - 1\rangle$, so at $t = 1$ the velocity is $\langle 2,\ 2\rangle$ and the speed is $\sqrt{2^2 + 2^2} = 2\sqrt{2}$. Its position at $t = 2$ is $\langle 4,\ 6\rangle$.

Defining Polar Coordinates and Differentiating in Polar Form

Polar coordinates 极坐标 locate a point by its distance r from the origin and angle θ: convert with $x = r\cos\theta$, $y = r\sin\theta$. A polar curve $r = f(\theta)$ is a parametric curve in θ, so its slope is

$$\frac{dy}{dx} = \frac{dy/d\theta}{dx/d\theta}, \quad \text{with } x = r\cos\theta,\ y = r\sin\theta.$$



Polar coordinates give a point by its distance r and angle theta

Finding the Area of a Polar Region

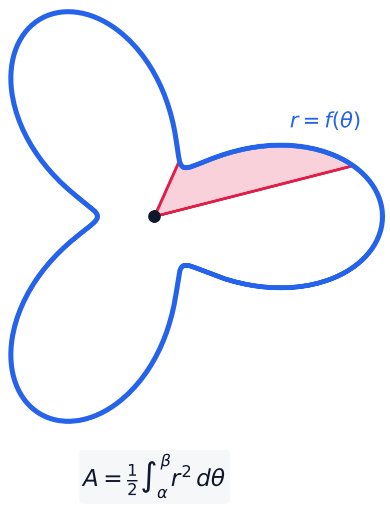
The area swept out by a polar curve $r = f(\theta)$ from α to β is

$$A = \frac{1}{2}\int_{\alpha}^{\beta} \big(f(\theta)\big)^2\, d\theta.$$

The region is a "fan" of thin triangular sectors; choosing the correct θ-limits (where the curve starts and finishes tracing the region) is the main challenge.

Worked example. Find the area of one petal of the rose $r = 2\sin(2\theta)$ (traced for θ from 0 to $\frac{\pi}{2}$). Using $\sin^2 u = \frac{1}{2}(1 - \cos 2u)$,

$$A = \frac{1}{2}\int_0^{\pi/2} (2\sin 2\theta)^2\, d\theta = \int_0^{\pi/2} (1 - \cos 4\theta)\, d\theta = \left[\theta - \tfrac{\sin 4\theta}{4}\right]_0^{\pi/2} = \frac{\pi}{2}.$$



The shaded region is a fan of thin sectors swept from α to β; each has area $\frac{1}{2}r^2\, d\theta$, so the total is $\frac{1}{2}\int_{\alpha}^{\beta} r^2\, d\theta$.

Finding the Area of the Region Bounded by Two Polar Curves

For the area between an outer curve r_1 and an inner curve r_2, subtract the sectors:

$$A = \frac{1}{2}\int_{\alpha}^{\beta} \left(r_1^2 - r_2^2\right) d\theta.$$

Find the **intersection angles** first (set $r_1 = r_2$), and be careful which curve is outer over each interval – they can swap.

Exam tips

- For **parametric** curves, $\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$; speed is $\sqrt{(dx/dt)^2 + (dy/dt)^2}$.
- A **vector-valued** function carries the same information — differentiate/integrate it component by component.
- In **polar**, convert with $x = r\cos\theta$, $y = r\sin\theta$; area swept is $\frac{1}{2}\int r^2\, d\theta$.
- Watch the direction of tracing (the sign of dx/dt) and set correct θ-limits for polar area.
- Eliminate the parameter to recover the ordinary y-vs-x shape when it helps.