Differential Equations

AP Calculus BC

Modeling Situations with Differential Equations

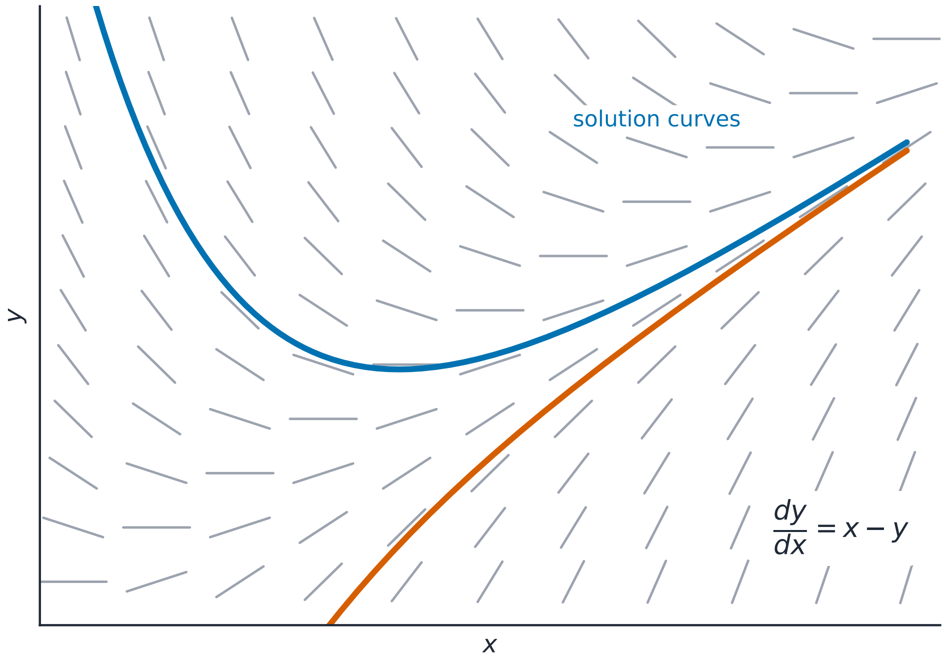
A **differential equation** 微分方程 relates a function to its **derivatives**. Many real situations are described by a rate: "the population grows at a rate proportional to its size" becomes $\frac{dP}{dt} = kP$. Setting up the equation from a verbal description – identifying what changes and what it is proportional to – is the first skill.

Verifying Solutions for Differential Equations

A **solution** is a function that satisfies the equation. To **verify** a proposed solution, differentiate it and substitute into the equation, checking that both sides agree. A **general solution** contains a constant C; a **particular solution** fixes C from a condition.

Sketching Slope Fields

A **slope field** 斜率场 draws a short line segment at many points, each with the slope $\frac{dy}{dx}$ the equation gives there. It pictures the family of solution curves without solving. To sketch one, evaluate the right-hand side at each grid point and draw a segment of that slope.



A slope field shows the gradient everywhere; solution curves follow it

Reasoning Using Slope Fields

A solution curve **follows the segments** like a boat following a current. From a slope field you can sketch the particular solution through a given point, describe long-run behavior, and locate where solutions level off (slopes near zero) – reasoning about solutions purely from the picture.

Approximating Solutions Using Euler's Method

Euler's method 欧拉方法 approximates a solution numerically by stepping along the slope field. Starting from a known point, take a small step Δx and update:

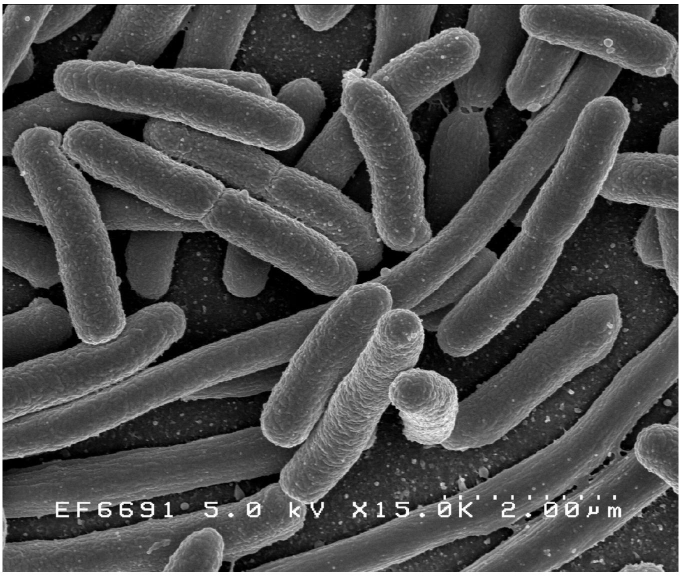
$$y_{\text{new}} = y_{\text{old}} + \frac{dy}{dx} \cdot \Delta x.$$

Repeat for each step. Smaller steps give a better approximation. (This is a BC-only technique.)

Exam skill: be able to carry out two or three Euler steps by hand from a table, and know that Euler's method **under- or over-estimates** depending on the solution's concavity.

Worked example. Approximate $y(1)$ for $\frac{dy}{dx} = x + y$, $y(0) = 1$, with step $\Delta x = 0.5$. Step 1: slope at $(0, 1)$ is $0 + 1 = 1$, so $y(0.5) \approx 1 + 1(0.5) = 1.5$. Step 2: slope at $(0.5, 1.5)$ is $0.5 + 1.5 = 2$, so $y(1) \approx 1.5 + 2(0.5) = 2.5$.

Finding General Solutions Using Separation of Variables



E. coli under a microscope: differential equations model exponential growth

A **separable** 可分离 differential equation can be written with all the y's on one side and all the x's on the other, then integrated:

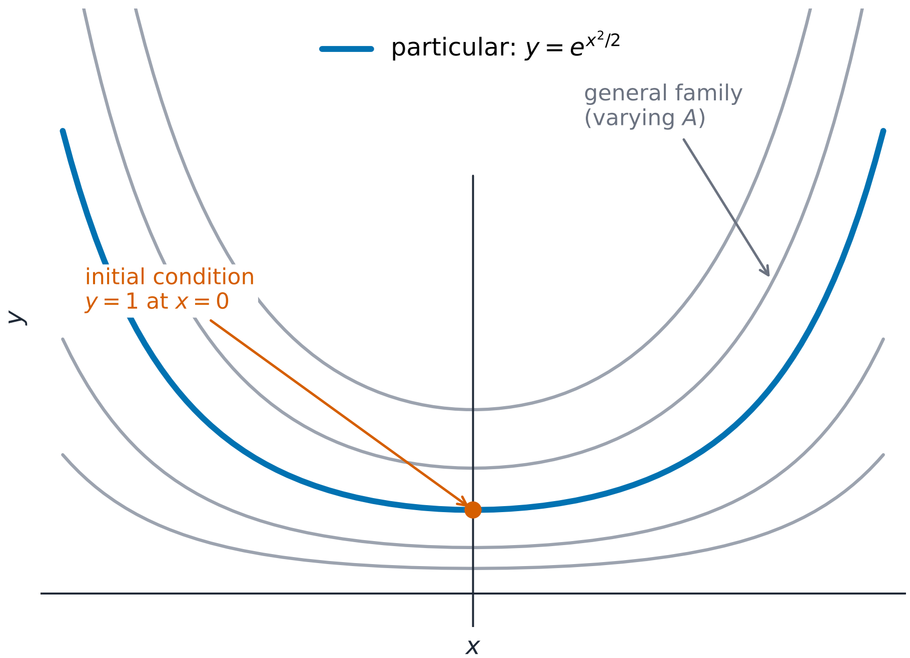
$$\frac{dy}{dx} = g(x)h(y) \;\Rightarrow\; \int \frac{dy}{h(y)} = \int g(x)\,dx.$$

This produces the general solution (with a $+C$), the main analytic method for solving differential equations in this course.

Worked example. Solve $\frac{dy}{dx} = xy$ with $y(0) = 2$. Separating, $\int \frac{dy}{y} = \int x\,dx$ gives $\ln|y| = \frac{x^2}{2} + C$, so $y = Ae^{x^2/2}$. The condition $y(0) = 2$ gives $A = 2$, so $y = 2e^{x^2/2}$.

Finding Particular Solutions Using Initial Conditions

An **initial condition** 初始条件 (a known point, e.g. $y(0) = 5$) pins down the constant C. Solve for the general solution, substitute the condition to find C, then write the particular solution. Watch the domain – a particular solution is valid only on the interval containing the initial point.



The constant gives a family of curves; an initial condition picks out one

Exponential Models with Differential Equations

The equation $\dfrac{dy}{dt} = ky$ says the rate of change is **proportional** to the amount – giving **exponential growth or decay 指数增长**. Separating variables yields

$$y = y_0 e^{kt},$$

with $k > 0$ for growth and $k < 0$ for decay. This models unrestricted population growth, radioactive decay, and continuously compounded interest.

A cooling cup of coffee: Newton's law of cooling is a classic differential equation model

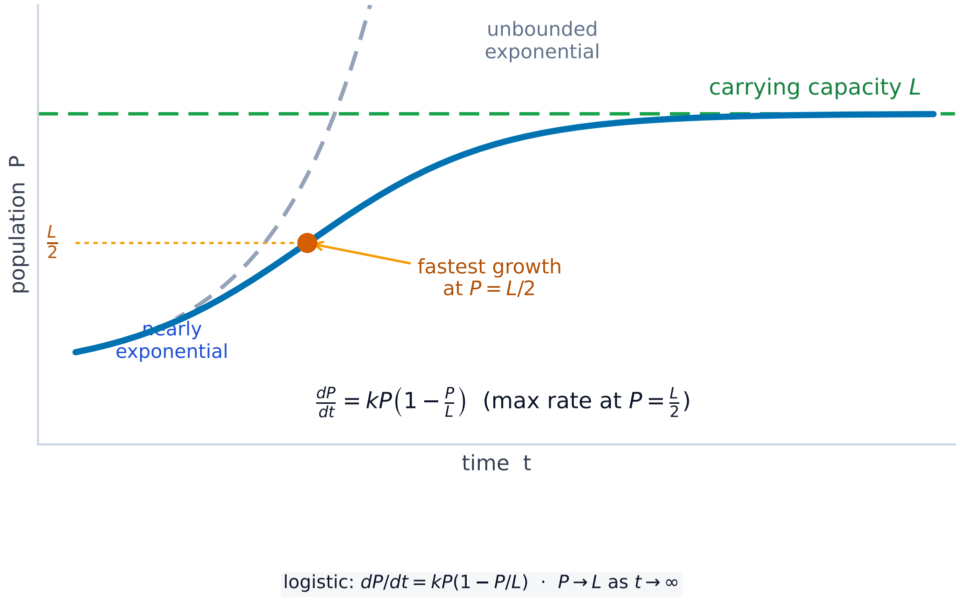
Logistic Models with Differential Equations

Real growth is limited by resources, so the **logistic model 逻辑斯蒂模型** adds a **carrying capacity 环境容纳量** L:

$$\frac{dP}{dt} = kP\left(1 - \frac{P}{L}\right).$$

Growth is nearly exponential when P is small, slows as P approaches L, and stops at $P = L$. Key BC facts: the population **levels off at** L ($\lim_{t\to\infty} P = L$), and it grows **fastest when** $P = \frac{L}{2}$ (the inflection point of the S-shaped curve). You are expected to read L and the fastest-growth value directly from the equation.

Worked example. For $\dfrac{dP}{dt} = 0.05\,P\left(1 - \dfrac{P}{2000}\right)$, the carrying capacity is $L = 2000$ (the population levels off there), and growth is fastest when $P = \dfrac{L}{2} = 1000$ – both read straight off the equation, no solving needed.



The logistic model grows fastest at P=L/2 and levels off at the carrying capacity L

Exam tips

- Solve a **separable** equation by getting all y on one side and all x on the other, then integrating both sides (add $+C$ once).
- Use the initial condition to find C (a particular solution).
- Sketch or read a **slope field**: the little segments show $\frac{dy}{dx}$ at each point, and a solution curve follows them.
- Recognise **exponential** models $\frac{dy}{dt} = ky \Rightarrow y = Ce^{kt}$ (growth/decay).
- A differential equation gives the slope — you must integrate to recover the function.