Analytical Applications of Differentiation

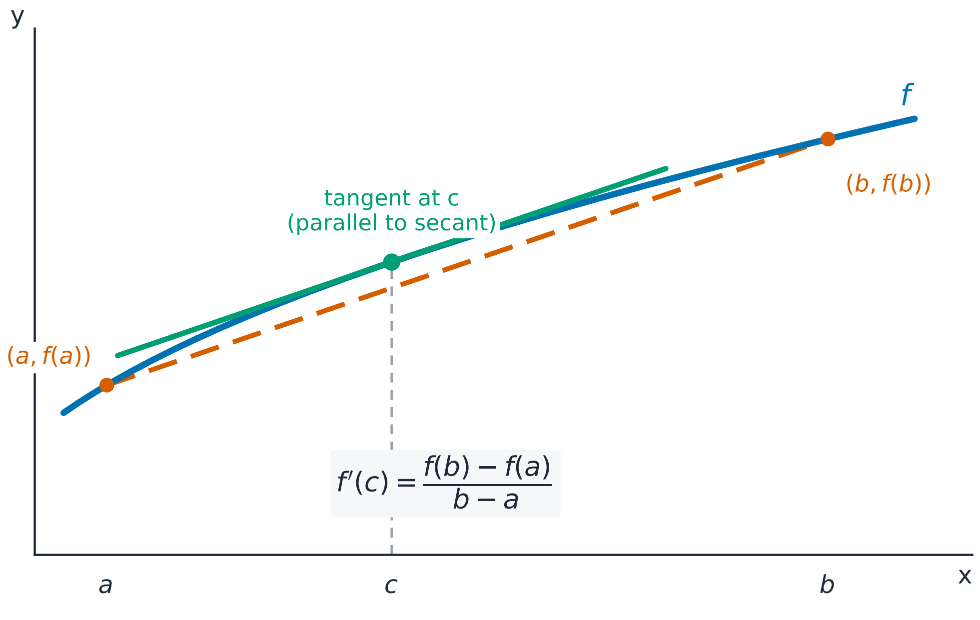
AP Calculus BC

Using the Mean Value Theorem

The **Mean Value Theorem** 中值定理 (MVT) links the average rate of change to an instantaneous one:

> If f is **continuous** on $[a, b]$ and **differentiable** on (a, b), then there is at least one point c in (a, b) where
>
> $$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

In words: somewhere inside the interval, the instantaneous rate equals the average rate. Geometrically, some tangent line is parallel to the line joining the endpoints.



The Mean Value Theorem: some tangent is parallel to the secant over the interval

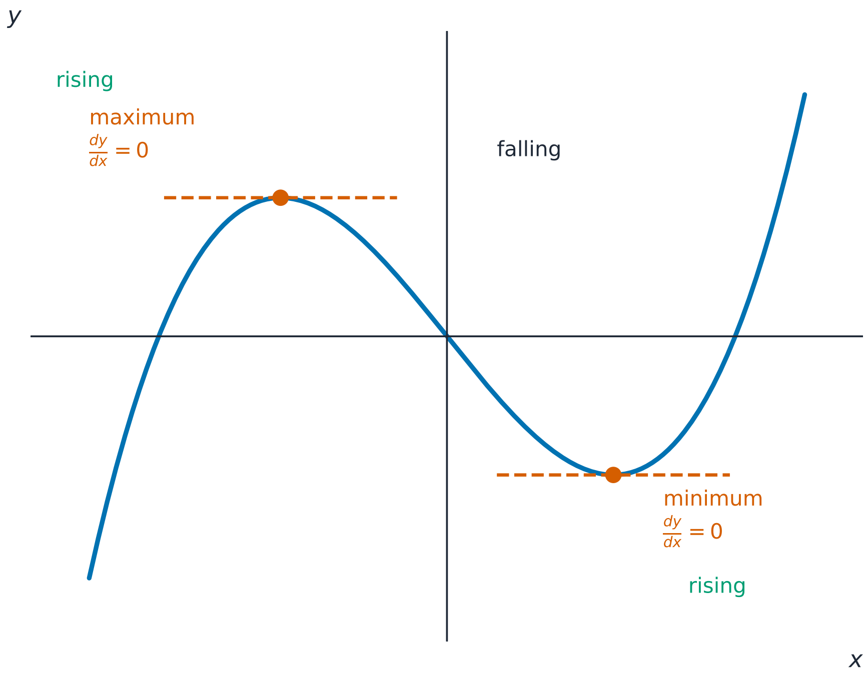
Exam skill. Like the IVT, the MVT is an existence theorem, and questions ask you to justify. Full credit needs: (1) state f is continuous on $[a, b]$ and differentiable on (a, b); (2) compute the average rate $\frac{f(b)-f(a)}{b-a}$; (3) conclude "by the MVT there is a c in (a, b) with $f'(c)$ equal to that value." Both hypotheses must be named.

Worked example. For $f(x) = x^2$ on $[1, 3]$ the average rate is $\dfrac{9-1}{2} = 4$; setting $f'(c) = 2c = 4$ gives $c = 2$, which lies in $(1, 3)$ – the guaranteed point.

Extreme Values, Global vs Local Extrema, and Critical Points

A mountain peak: extrema and critical points mark highest and lowest values

The **Extreme Value Theorem** 极值定理 (EVT) guarantees extremes exist: a function **continuous** on a closed interval $[a, b]$ attains **both** an absolute maximum and an absolute minimum on it.



At a maximum or minimum the derivative is zero

A **critical point** 临界点 is an interior point where $f'(x) = 0$ **or** $f'(x)$ does not exist. All **local (relative) extrema** 局部极值 occur at critical points – but **not every critical point is an extremum**. So critical points are the *candidates*; you must test each.

Where a Function Increases or Decreases

The **first derivative** tells you where f rises or falls:

- $f'(x) > 0$ on an interval $\Rightarrow f$ is **increasing** 递增 there;
- $f'(x) < 0 \Rightarrow f$ is **decreasing** 递减.

On the exam, "find the intervals where f is increasing" means: find the critical points, then test the sign of f' between them, and **justify with the sign of** f' (a stated reason, not just an interval).

The First Derivative Test for Local Extrema

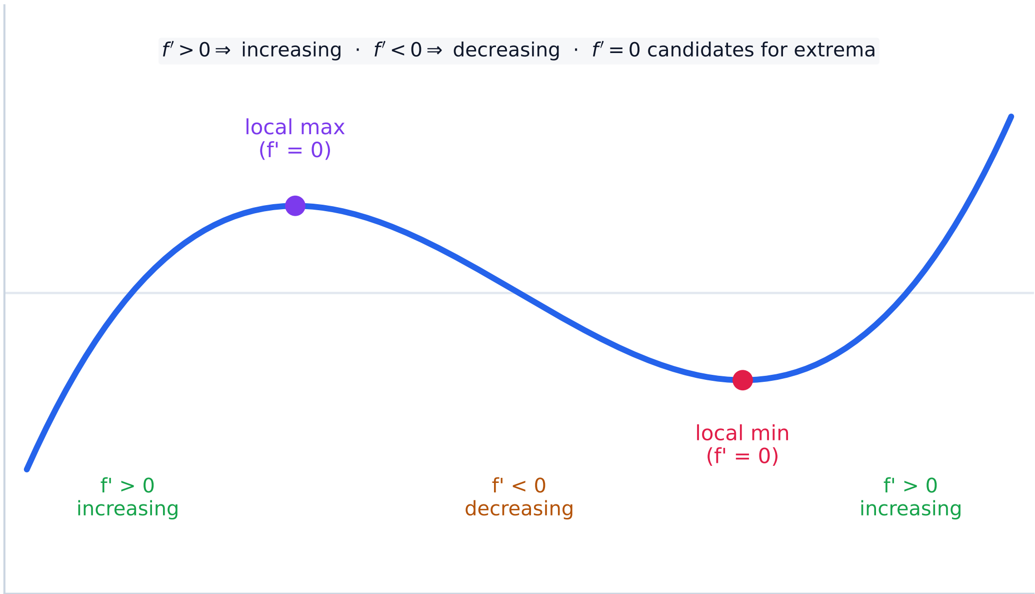
To classify a critical point $x = c$ as a local max, local min, or neither, check how f' changes sign there:

- f' changes $+$ **to** $-$ at $c \Rightarrow$ **local maximum** 极大值;

- f' changes $-$ **to** $+$ at $c \Rightarrow$ **local minimum 极小值**;
- f' does **not** change sign $\Rightarrow$ neither.

Always state the sign change as your justification.

Worked example. For $f(x) = x^3 - 3x^2$, $f'(x) = 3x(x-2)$ is zero at $x = 0, 2$. Signs give $+, -, +$, so $x = 0$ is a **local maximum** ($f = 0$) and $x = 2$ a **local minimum** ($f = -4$).



Where $f' > 0$ the graph rises and where $f' < 0$ it falls; a local max sits where f' turns $+$ to $-$, a local min where it turns $-$ to $+$.

The Candidates Test for Absolute Extrema

On a **closed interval**, absolute (global) extrema occur only at **critical points or endpoints**. The **candidates test**:

1. List all critical points in $[a, b]$ **and** the two endpoints.
2. Evaluate f at each candidate.
3. The largest output is the absolute maximum; the smallest is the absolute minimum.

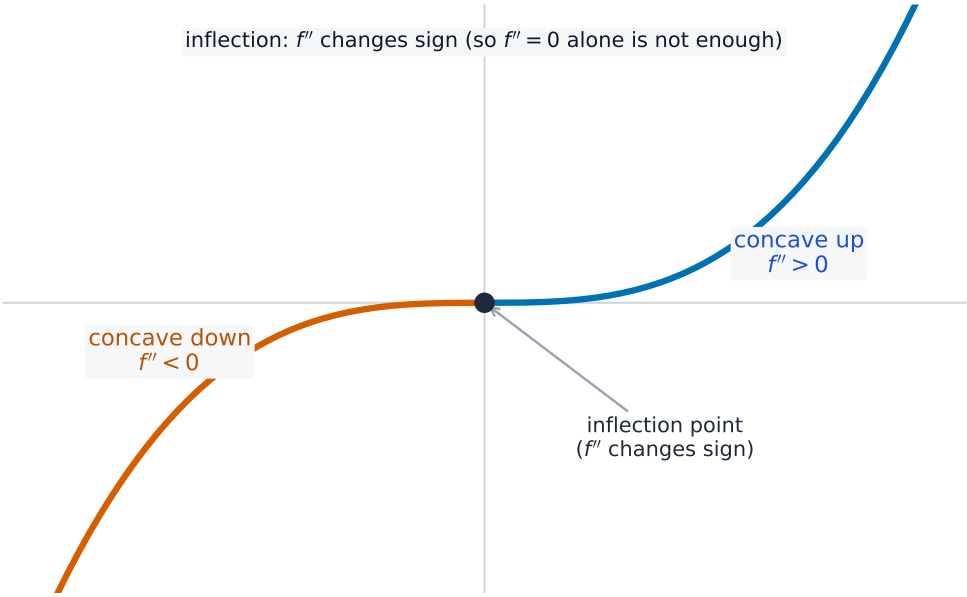
Show the table of values – the comparison is the argument.

Concavity and Points of Inflection

The **second derivative** describes bending:

- $f'' > 0 \Rightarrow f$ is **concave up** 上凹 (curving like a cup; f' is increasing);
- $f'' < 0 \Rightarrow f$ is **concave down** 下凹 (f' is decreasing).

A **point of inflection** 拐点 is where concavity **changes**, i.e. where f'' changes sign (not merely where $f'' = 0$). Report its x-coordinate and justify with the sign change of f''.



Concavity comes from the sign of the second derivative; it flips at an inflection point

The Second Derivative Test for Extrema

An alternative way to classify a critical point c where $f'(c) = 0$:

- $f''(c) > 0 \Rightarrow$ concave up $\Rightarrow$ **local minimum**;
- $f''(c) < 0 \Rightarrow$ concave down $\Rightarrow$ **local maximum**;
- $f''(c) = 0 \Rightarrow$ the test is **inconclusive** – fall back on the first derivative test.

Special case: if a continuous function has only **one** critical point on an interval and it is a local extremum, that point is also the **absolute** extremum there.

Sketching a Function and Its Derivative

Key features of f, f', and f'' mirror each other. To sketch or read graphs:

- f increasing $\Leftrightarrow$ f' above the axis; f has a local max $\Leftrightarrow$ f' crosses from $+$ to $-$.
- f concave up $\Leftrightarrow$ f' increasing $\Leftrightarrow$ f'' above the axis; f has an inflection point $\Leftrightarrow$ f' has a local extremum $\Leftrightarrow$ f'' crosses zero.

Connecting f, f', and f''

This is the skill of reading one graph to describe another. A very common exam setup gives the graph of f' and asks about f: where is f increasing (where $f' > 0$), where are f's extrema (where f' crosses zero, with a sign change), where is f concave up (where f' is increasing). Answer questions about f using the *height and slope* of the f' graph.

Introduction to Optimization Problems

Optimization 最优化 uses the derivative to find the largest or smallest value of a quantity on an interval. It is the candidates/derivative-test machinery applied to a real goal.

Fenced enclosures: optimization finds the dimensions that maximise area for a fixed fence length

Solving Optimization Problems

A dependable procedure:

1. Write the quantity to optimize as a function of **one** variable (use a constraint equation to eliminate extras).
2. State the interval of allowed inputs.
3. Find critical points ($f' = 0$ or undefined) and test them (first- or second-derivative test, or candidates test if the interval is closed).
4. Answer the question asked, **with units and interpretation** in context – the maximum *area*, the minimum *cost*, etc.

Worked example. With 100 m of fence for a rectangular pen against a wall (only three sides fenced), let the ends be x and the far side $y = 100 - 2x$. The area $A(x) = x(100 - 2x) = 100x - 2x^2$ has $A'(x) = 100 - 4x = 0$ at $x = 25$; since $A'' = -4 < 0$ this is the maximum, giving $y = 50$ and $A = 1250\ \mathrm{m}^2$.

Exploring Behaviors of Implicit Relations

All of this extends to **implicitly defined** relations. A critical point of an implicit relation is where $\dfrac{dy}{dx} = 0$ (horizontal tangent) or is undefined (vertical tangent). Because $\dfrac{dy}{dx}$ is usually a relation in x **and** y, and the second derivative involves x, y, and $\dfrac{dy}{dx}$, substitute your first-derivative expression back in when finding $\dfrac{d^2y}{dx^2}$, then reason about concavity from its sign.

Exam tips

- $f' > 0$ means increasing, $f' < 0$ decreasing; candidates for **extrema** are where $f' = 0$ or is undefined.
- Classify a critical point with the **first-derivative sign change** or the **second-derivative test** ($f'' > 0$ minimum, $f'' < 0$ maximum).
- $f'' > 0$ is concave up, $f'' < 0$ concave down; a **point of inflection** is where concavity changes (f'' changes sign).
- For an **absolute** extremum on a closed interval, also check the endpoints.
- Justify every conclusion by citing the sign of f' or f'' — the exam demands the reasoning, not just the answer.