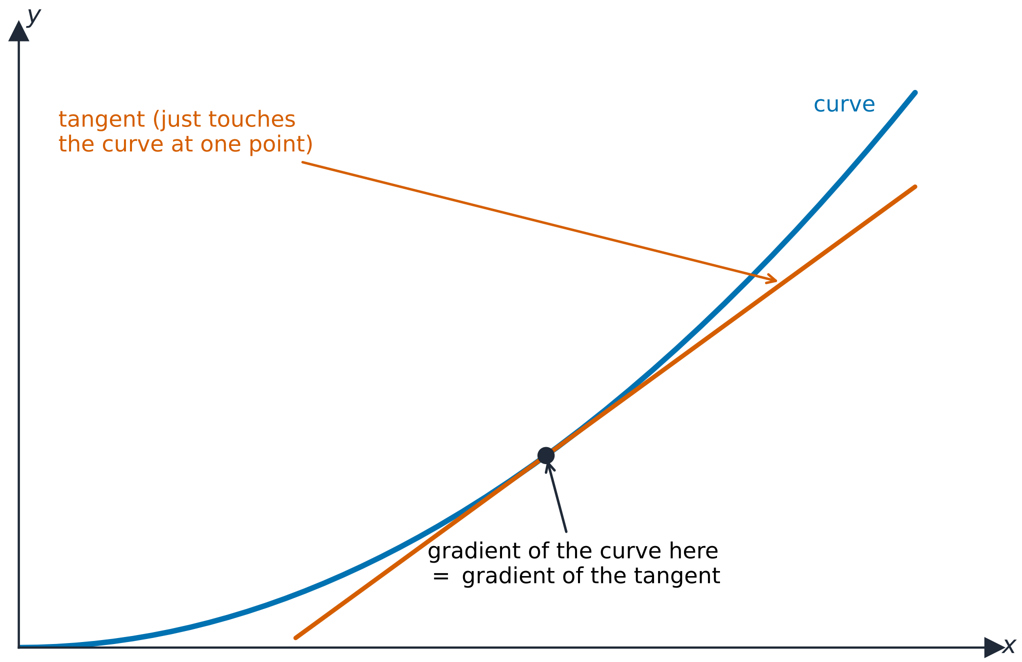
Differentiation: Definition and Fundamental Properties

AP Calculus AB

Average and Instantaneous Rates of Change

A roller coaster on a lift hill: the derivative measures instantaneous rate of change of position

Unit 1 built the limit. Unit 2 uses it to define the **derivative** 导数 – the exact rate of change at a point.



The instantaneous rate of change is the gradient of the tangent at a point

Over an interval, the average rate of change is a **difference quotient** 差商. Two equivalent forms appear:

$$\frac{f(a+h)-f(a)}{h} \qquad \text{and} \qquad \frac{f(x)-f(a)}{x-a}.$$

The first uses a step of size h from a; the second uses two points x and a. Both compute $\dfrac{\text{change in output}}{\text{change in input}}$ over the interval.

The **instantaneous** 瞬时 rate of change at $x = a$ is what the difference quotient approaches as the interval shrinks to zero. This limit *is* the derivative at a, written $f'(a)$:

$$f'(a) = \lim_{h\to 0} \frac{f(a+h)-f(a)}{h} = \lim_{x\to a} \frac{f(x)-f(a)}{x-a},$$

provided the limit exists.

Defining the Derivative and Its Notation

Let the point a vary and the derivative becomes a **new function**:

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}.$$

This is the **definition of the derivative** (sometimes called differentiating "by first principles" 用定义求导). Its value at each x is the instantaneous rate of change there.

Common **notations** 记号 for the derivative of $y = f(x)$ are:

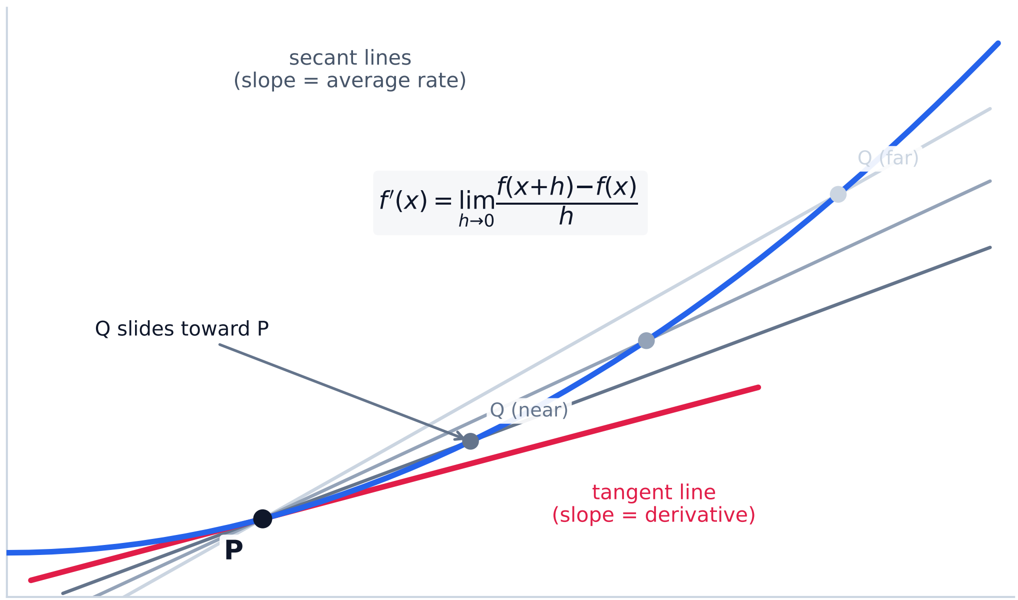
$$\frac{dy}{dx}, \qquad f'(x), \qquad y'.$$

The derivative can be represented graphically, numerically, analytically, and verbally – be ready to move between them.

Geometric meaning. The derivative at a point is the **slope** 斜率 of the **tangent line** 切线 to the graph there. So the tangent line at $x = a$ passes through $\big(a, f(a)\big)$ with slope $f'(a)$:

$$y - f(a) = f'(a)\,(x - a).$$

Writing this line is a routine exam task, so keep the point-slope form ready.



As Q slides toward P, each secant's slope (an average rate) approaches the tangent's slope – the derivative $f'(a)$.

Estimating a Derivative at a Point

You do not always have a formula. When a function is given by a **table** 表格 or a graph, estimate the derivative $f'(a)$ with a difference quotient over a **small interval around** a. A table with values on both sides of a gives the best estimate:

$$f'(a) \approx \frac{f(b) - f(c)}{b - c}, \qquad \text{where } c < a < b \text{ are the closest table inputs.}$$

Technology (a calculator) can also estimate a derivative at a point.

Exam skill (appears almost every year). Questions such as "Approximate $M'(7.5)$ using the average rate of change of M over the interval $5 \le t \le 10$" ask for exactly this difference quotient. Show the setup:

$$M'(7.5) \approx \frac{M(10) - M(5)}{10 - 5}.$$

Full credit needs the numbers plugged in **and** the correct **units** 单位 (output units per input unit), since these come from real-world models.

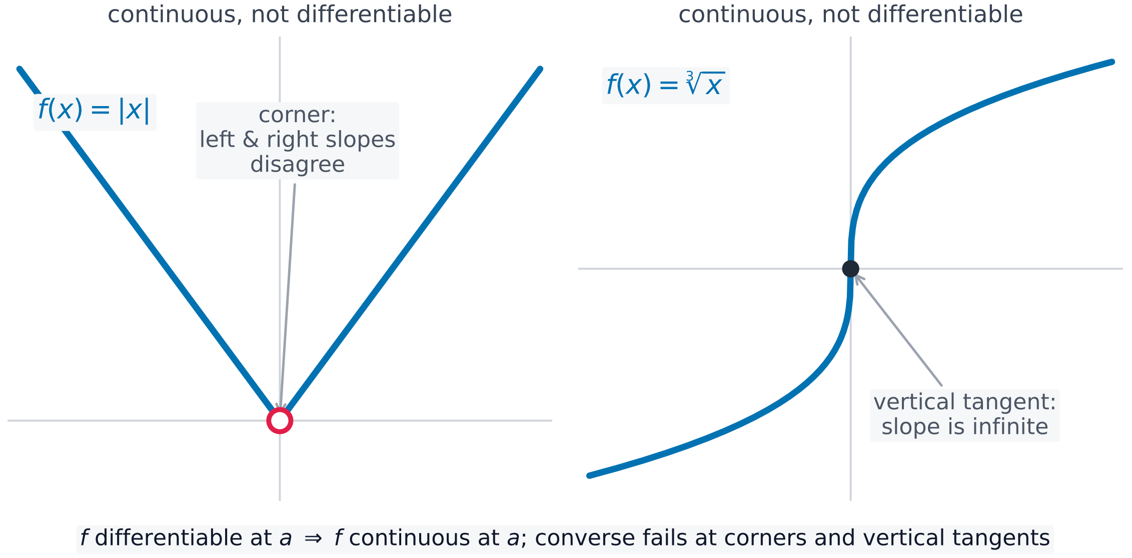
When Does a Derivative Exist?

Differentiability is stronger than continuity. The key relationship:

> **If f is differentiable 可导 at a point, then f is continuous 连续 there.**

So differentiability *implies* continuity. The reverse is false: a continuous function can fail to be differentiable. Two ways this happens:

- A **corner** 尖点: the left and right difference-quotient limits disagree, as with $f(x) = |x|$ at $x = 0$.
- A **vertical tangent** 垂直切线: the slope is infinite (no real number), as with $f(x) = \sqrt[3]{x}$ at $x = 0$.



Two ways a continuous function is not differentiable: a corner and a vertical tangent

Also, a point outside the domain of f cannot be in the domain of f'. Use the contrapositive on the exam: **if f is not continuous at a, then f is not differentiable at a.**

The Power Rule

From here we use **rules** instead of the limit definition each time. The **power rule** 幂法则 handles any power of x:

$$\frac{d}{dx}\,x^r = r\,x^{\,r-1} \qquad \text{for any real } r.$$

It works for whole-number powers, negative powers ($\frac{1}{x} = x^{-1}$), and roots ($\sqrt{x} = x^{1/2}$) – rewrite as a power first, then apply the rule.

Constant, Sum, Difference, and Constant Multiple Rules

These rules let you differentiate term by term:

- **Constant:** $\dfrac{d}{dx}\,k = 0$ (a constant does not change).
- **Constant multiple** 常数倍: $\dfrac{d}{dx}\big[k\,f(x)\big] = k\,f'(x)$.
- **Sum / difference:** $\dfrac{d}{dx}\big[f(x) \pm g(x)\big] = f'(x) \pm g'(x)$.

Combined with the power rule, they differentiate any **polynomial** 多项式 term by term. Example:

$$\frac{d}{dx}\big(4x^3 - 5x + 7\big) = 12x^2 - 5.$$

Derivatives of sin x, cos x, e^x, and ln x

Learn these four building-block derivatives by heart:

$$\frac{d}{dx}\sin x = \cos x, \qquad \frac{d}{dx}\cos x = -\sin x,$$

$$\frac{d}{dx}e^x = e^x, \qquad \frac{d}{dx}\ln x = \frac{1}{x} \quad (x > 0).$$

Note the minus sign on the derivative of cosine, and that e^x is its own derivative.

A limit that is really a derivative (LIM-3.A.1). Sometimes a limit is secretly the definition of a known derivative. If you recognize

$$\lim_{h\to 0} \frac{f(a+h)-f(a)}{h}$$

for a function f whose derivative you know, just evaluate $f'(a)$. For example, $\lim_{h\to 0} \frac{\sin\left(\frac{\pi}{2}+h\right)-1}{h} = \frac{d}{dx}\sin x\Big|_{x=\pi/2} = \cos\frac{\pi}{2} = 0.$

The Product Rule

A product of two functions is **not** differentiated by multiplying the derivatives. Use the **product rule** 乘积法则:

$$\frac{d}{dx}[u\,v] = u'v + uv'.$$

"Derivative of the first times the second, plus the first times the derivative of the second." Example:

$$\frac{d}{dx}\left(x^2 e^x\right) = 2x\,e^x + x^2 e^x.$$

Exam questions often build a new function from given pieces, e.g. $k'(x) = \big(f(x)\big)^2 g(x)$, and ask you to combine rules while reading values from a table.

The Quotient Rule

For a quotient, use the **quotient rule** 商法则:

$$\frac{d}{dx}\left[\frac{u}{v}\right] = \frac{u'v - uv'}{v^2}.$$

"Bottom times derivative of top, minus top times derivative of bottom, all over bottom squared." The order matters because of the minus sign, so write the numerator carefully. Example:

$$\frac{d}{dx}\left(\frac{x}{\cos x}\right) = \frac{1\cdot\cos x - x\cdot(-\sin x)}{\cos^2 x} = \frac{\cos x + x\sin x}{\cos^2 x}.$$

Derivatives of tan x, cot x, sec x, and csc x

The remaining trigonometric derivatives are not memorized separately – you rewrite them with **identities** 恒等式 and apply the quotient (or product) rule. For instance, $\tan x = \dfrac{\sin x}{\cos x}$, so the quotient rule gives

$$\frac{d}{dx}\tan x = \frac{\cos x \cos x - \sin x(-\sin x)}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x.$$

The same method (writing $\cot x = \frac{\cos x}{\sin x}$, $\sec x = \frac{1}{\cos x}$, $\csc x = \frac{1}{\sin x}$) gives $-\csc^2 x$, $\sec x \tan x$, and $-\csc x \cot x$.

Higher-order derivatives. Differentiating f' again gives the **second derivative** 二阶导数 $f''(x)$ (or $\frac{d^2y}{dx^2}$) – the rate of change of the rate of change. An exam part like "Find $k''(3)$" just means differentiate twice, then substitute. You can also estimate a second derivative from a table by applying the average-rate-of-change method to the f' values.

Worked example. Differentiate $f(x) = x^2 \sin x$ with the **product rule** $(uv)' = u'v + uv'$: take $u = x^2$ ($u' = 2x$) and $v = \sin x$ ($v' = \cos x$), giving $f'(x) = 2x \sin x + x^2 \cos x$. Always read the *structure* first — a product needs the product rule, not the power rule applied to each factor separately.

Exam tips

- The derivative is the **slope of the tangent** — the limit of the secant slope $\frac{f(x+h)-f(x)}{h}$ as $h \to 0$.
- Memorise the rules: power, product, quotient, and the derivatives of sin, cos, e^x, and $\ln x$.
- **Differentiability implies continuity**, but not the reverse (a corner or cusp is continuous yet not differentiable).
- Distinguish average rate of change (secant slope over an interval) from instantaneous rate (the derivative at a point).
- Give a tangent-line equation as $y - f(a) = f'(a)(x - a)$.