Forces, density and pressure

A-Level Physics

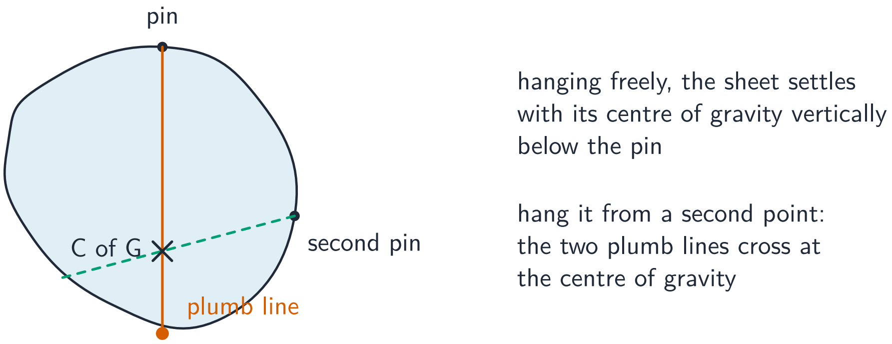
Turning effects of forces

Centre of gravity

The **weight** 重力 of a large object can be treated as acting at one single point, called the **centre of gravity** 重心 (the same as the **centre of mass** 质心 in a uniform gravitational field). For a uniform, regular shape — a rectangle, a sphere, a uniform rod — the centre of gravity is at the middle.

When you draw a **free-body diagram** 受力图, always put the weight arrow at the centre of gravity.

For an irregular shape, find the centre of gravity by experiment. Hang the object freely from a pin: in equilibrium it settles with its centre of gravity **vertically below the pin**, because only then does the weight have no moment about the pin. A **plumb line** 铅垂线 hung from the same pin marks that vertical. Hang the object from a second point and draw the second line; the centre of gravity is where the two lines cross. The same fact answers the exam's "explain why the sheet hangs like this": its centre of gravity is directly below the support, so the weight's line of action passes through the pivot and there is no resultant torque.



Finding a centre of gravity: hung freely, an object settles with its centre of gravity vertically below the pin, so two plumb lines from two pins cross at it

For a **non-uniform** 非均匀 bar, the centre of gravity is not at the middle. Find it by balancing the bar on a pivot, or by taking moments: if the bar's weight W balances a known weight, the distance of its centre of gravity from the pivot is the one moment that makes the two sides equal.

Moment of a force

The **moment** 力矩 of a force about a point is

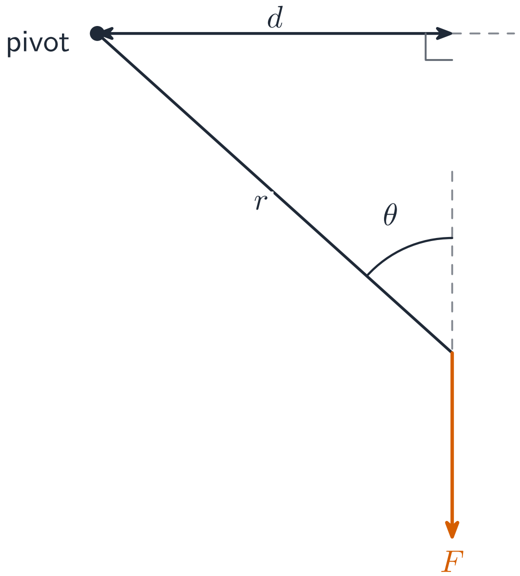
$$M = F \cdot d,$$

where F is the size of the **force** 力 and d is the **perpendicular distance** 垂直距离 from the point to the **line of action** 作用线 of the force. Unit: N m.

If the force acts at angle θ to a **lever arm** 力臂 of length r from the **pivot** 支点, then $d = r \sin\theta$, so $M = Fr\sin\theta$. Only the part of the force **perpendicular** 垂直 to the lever arm makes it turn.

Worked example. A force of 50 N is applied to the end of a spanner 0.40 m long, at 30° to the spanner. Find the moment of the force about the nut.

$$M = Fr\sin\theta = 50 \times 0.40 \times \sin 30° = 50 \times 0.40 \times 0.50 = 10 \text{ N m}.$$



The moment depends on the perpendicular distance d from the pivot to the line of action of the force

A moment is either **clockwise** 顺时针 or **anticlockwise** 逆时针 about the chosen point.

Couple and torque

A **couple** 力偶 is a pair of forces that are:

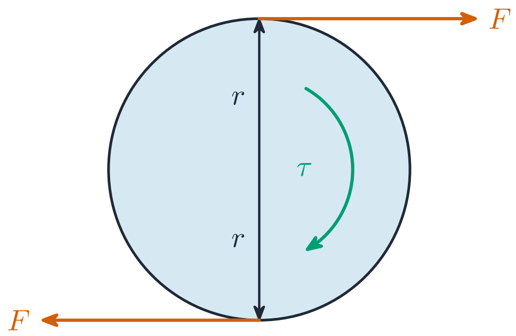
- **equal in size**,
- **opposite in direction**,
- with their lines of action a perpendicular distance apart.

A couple makes the body **turn only** — its resultant force is zero, so it gives no straight-line acceleration.

The **torque** 力偶矩 of a couple is the turning effect it makes:

$$\tau = F \cdot d,$$

where F is the size of one force and d is the perpendicular distance between the two lines of action. Unit: N m. The torque is the same about any point — a special property of couples.



A couple: two equal and opposite forces, a distance apart, producing a torque

A common multiple-choice trap: two equal forces in the **same** direction are not a couple (they have a resultant force and cause **translation** 平动). A couple needs equal size **and** opposite direction.

Watch the distance in the formula. For a couple of two forces F each acting a distance d from a central pivot, on opposite sides, the perpendicular distance between the two lines of action is $2d$, so the torque is $F \times 2d = 2Fd$. Taking moments about the pivot gives the same answer, $Fd + Fd$. The two-mark definition wants both parts: **the product of one of the forces and the perpendicular distance between the lines of action of the forces.**

Equilibrium of forces

Conditions for equilibrium

A body is in **equilibrium** 平衡 when:

1. the **resultant force** 合力 is zero (no straight-line acceleration), AND
2. the resultant moment about any point is zero (no **angular acceleration** 角加速度).

Both must hold. A body with no resultant force can still be turning; a body with no resultant moment can still be moving in a straight line.

The classic multiple-choice case: four forces, two equal pairs pointing in opposite directions but not along the same lines. The resultant force is zero, so the centre of mass does not accelerate, but the pairs form couples with a resultant torque, so the object spins faster and faster while its centre of mass moves at constant velocity. Only when both conditions hold is the body in equilibrium.

Principle of moments

For a body that is not turning, **the total clockwise moment about any point equals the total anticlockwise moment about the same point**. This is the **principle of moments** 力矩原理.

To solve a balance problem:

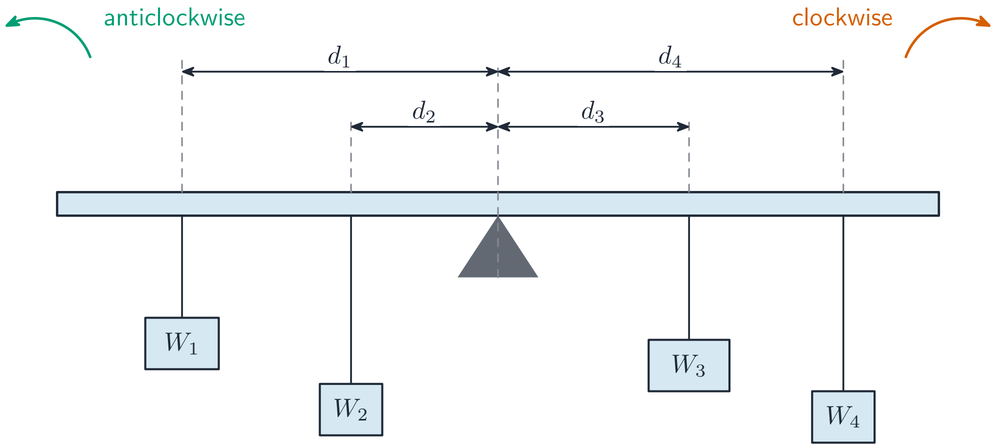
1. Choose a pivot — usually where an unknown force acts, so that force drops out (its distance is zero).
2. List every force and its perpendicular distance from the pivot.
3. Set $\sum M_{\text{clockwise}} = \sum M_{\text{anticlockwise}}$.
4. Use $\sum F = 0$ if you need a second equation.

A ruler balanced on a pivot with masses on each side is solved this way. For a heavy uniform rod, remember to include its weight acting at its centre of gravity. Stated for the marks: **for a body in equilibrium, the sum of the clockwise moments about any point equals the sum of the anticlockwise moments about that same point.**

Worked example. A uniform beam of length 4.0 m is pivoted at its centre. A 40 N weight hangs 1.5 m from the pivot on one side. How far from the pivot, on the other side, must a 30 N weight hang to balance the beam?

The beam's own weight acts at the centre (the pivot), so it has no moment. Setting clockwise = anticlockwise moments:

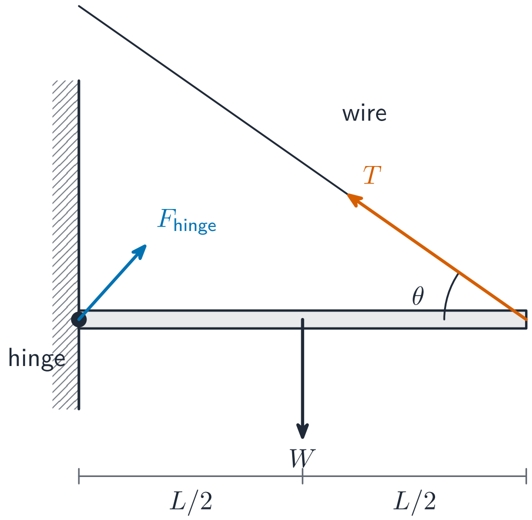
$$40 \times 1.5 = 30 \times d \quad \Rightarrow \quad d = \frac{60}{30} = 2.0 \text{ m.}$$



Weights on a rule balanced at a pivot — used to test the principle of moments

A tower crane is a moments problem: the counterweight's moment about the mast balances the moment of the load

Worked example. A uniform rod of length L and weight $W = 28$ N is hinged to a wall at one end and held horizontal by a wire from its other end. The wire makes 40° with the rod. Find the tension T in the wire.



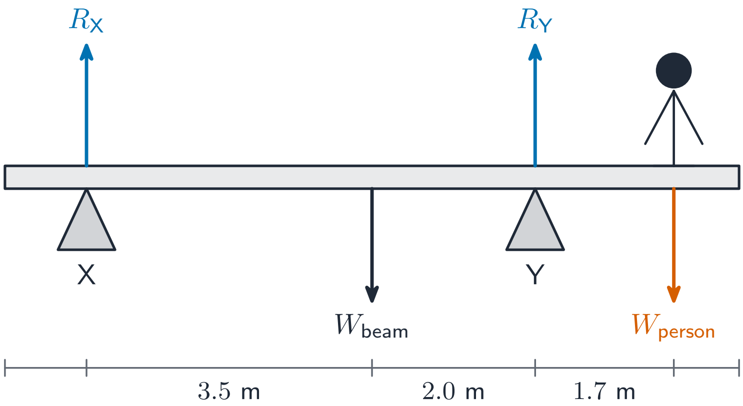
A hinged rod held by a wire: taking moments about the hinge removes the unknown hinge force

Take moments about the hinge, so the unknown hinge force drops out. The weight acts at the centre, a distance $L/2$ from the hinge: its moment is $W \times L/2$, clockwise. The tension acts at the end, but not at right angles to the rod, so its moment is $T \sin 40° \times L$, anticlockwise (only the component of T perpendicular to the rod turns it). Equate them:

$$T \sin 40° \times L = W \times \frac{L}{2} \quad \Rightarrow \quad T = \frac{28}{2 \sin 40°} = 22 \text{ N}.$$

The length L cancels, which is why the question need not give it. If a load hangs from the rod as well, add its moment on the clockwise side. This is the shape of almost every moments question: choose the pivot where the unknown force acts, use the perpendicular component of any angled force, and let unknown lengths cancel.

Worked example. A uniform beam of weight 200 N and length 9.0 m rests on two supports, X at 1.0 m from the left end and Y at 6.5 m. A person of weight 700 N stands 1.7 m from Y, beyond it. Find the force from each support.



A beam on two supports: moments about one support give the other support's force; the sum of forces then gives the first

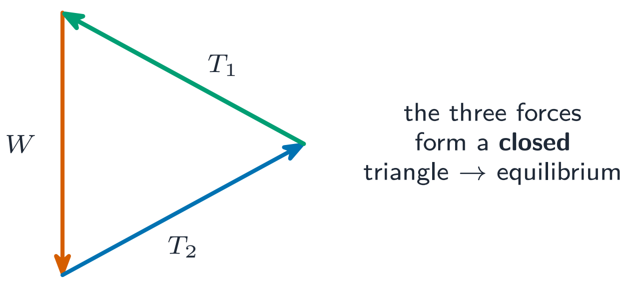
Two unknown forces, R_X and R_Y, so take moments about X to remove R_X. The beam's weight acts at its centre, 3.5 m from X; Y is 5.5 m from X; the person is 7.2 m from X:

$$R_Y \times 5.5 = 200 \times 3.5 + 700 \times 7.2 \quad \Rightarrow \quad R_Y = \frac{700 + 5040}{5.5} = 1040 \text{ N}.$$

Then the resultant force is zero: $R_X + R_Y = 200 + 700$, so $R_X = 900 - 1040 = -140$ N. A negative answer means the beam would lift off X: with the person that far past Y, X must hold the beam down, not up. Questions often ask where a person can stand before the beam tips; that is the position at which R_X becomes zero.

Vector triangle

Three forces in the same plane that are in equilibrium can be drawn as a **closed vector triangle** 矢量三角形 — drawn tip-to-tail, the three arrows come back to the start. This is a drawing method instead of splitting into **components** 分量.



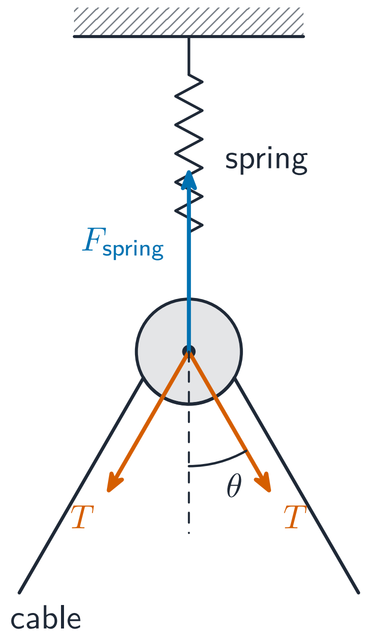
Three forces in equilibrium form a closed vector triangle

Use the **sine rule** 正弦定理 or the **cosine rule** 余弦定理 on the triangle to find unknown sizes or directions, or draw the triangle to scale on graph paper.

You can also split each force into **horizontal** 水平 and **vertical** 竖直 components and set $\sum F_x = 0$ and $\sum F_y = 0$.

To draw the triangle for the marks: draw the weight first (vertical, to scale), then add the other two forces tip to tail in their real directions, so the third arrow closes the triangle; label every side with its force and mark the angles. For a block resting on a slope the three forces are the weight (vertical), the normal contact force (perpendicular to the slope) and friction (along the slope), so the triangle is right-angled with the weight as the hypotenuse. For a picture hanging from a string over a pin, the two equal tensions make an isosceles triangle with the weight. The test of any answer is that the triangle closes; if it does not, the object is not in equilibrium.

Worked example. A pulley of negligible weight is held by a spring. A single cable passes under the pulley, and each side of the cable makes 30° with the vertical. The tension in the cable is $T = 60$ N. Find the force from the spring and the extension of the spring, given its spring constant is 2000 N m^{-1}.



A pulley held by a spring: the two tensions in the one cable are equal, and the spring force balances their vertical components

Because it is one continuous cable, the tension is the same on both sides. The two tensions pull down and outwards; their horizontal components cancel and their vertical components add, so the spring must pull up with $F = 2T\cos 30° = 2 \times 60 \times 0.866 =$

104 N. From Hooke's law (topic 6), $x = F/k = 104/2000 = 0.052$ m. If the angle to the vertical grows, $\cos\theta$ falls, so the spring force and extension fall: the cable is pulling more sideways and less upwards.

Density

An iceberg floats with most of its volume hidden: ice is slightly less dense than water.

Density 密度 is the mass per unit volume:

$$\rho = \frac{m}{V}.$$

Unit: kg m^{-3} (or g cm^{-3}; 1 g cm^{-3} = 1000 kg m^{-3}). Density is a **scalar 标量**.

Some useful densities to know:

- water: 1000 kg m^{-3}
- air at room conditions: $\sim$ 1.2 kg m^{-3}
- iron / steel: $\sim$ 7800 kg m^{-3}

Measuring density is a practical question in disguise. For a cuboid, measure the mass on a balance and the three side lengths with a rule or calipers, then $\rho = m/(xyz)$. Because density is a product and quotient, the percentage uncertainty in ρ is the **sum** of the percentage uncertainties in m, x, y and z (topic 1). A zero error on the balance, or a rule whose end is worn, is a systematic error; repeating the readings does not remove it, but subtracting the zero reading does.

Worked example. A block has mass (120 ± 1) g and sides (4.0 ± 0.1) cm, (3.0 ± 0.1) cm and (2.0 ± 0.1) cm. Find its density and the percentage uncertainty.

$\rho = 120/(4.0 \times 3.0 \times 2.0) = 5.0$ g cm^{-3} = 5000 kg m^{-3}. The percentage uncertainties are 0.8%, 2.5%, 3.3% and 5.0%, so the density is uncertain by 12%, about $\pm$600 kg m^{-3}.

The shortest side contributes most, which is why a question asks which measurement to improve first.

Pressure

Pressure 压强 is the force per unit area, where the force acts at right angles to the surface:

$$p = \frac{F}{A}.$$

Unit: $\mathrm{Pa} = \mathrm{N\ m^{-2}}$. Pressure is a scalar.

The force is the one pressing at right angles to the surface, and the area is the area of contact. A tree trunk of weight 9.0 kN standing on a post of diameter 0.28 m presses with $p = F/A = 9000/(\pi \times 0.14^2) = 1.5 \times 10^5$ Pa; the same weight on a wider post gives a smaller pressure. Convert diameters to radii, and centimetres to metres, before squaring.

A precision aneroid barometer measures atmospheric pressure

Hydrostatic pressure

Take a column of **fluid** 流体 with density ρ, **cross-sectional area** 横截面积 A and height Δh. Its weight is

$$W = mg = (\rho \cdot A \cdot \Delta h) \cdot g.$$

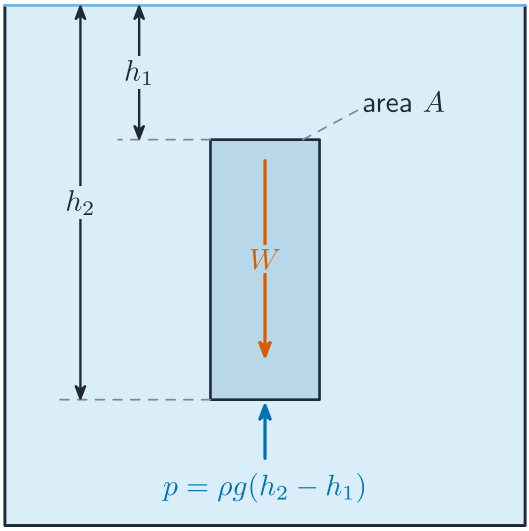
This weight presses down on the area A at the bottom, so the extra pressure at the bottom compared with the top is

$$\Delta p = \frac{W}{A} = \rho g \Delta h.$$

This is the **hydrostatic pressure 流体静压强** equation. It depends only on the density of the fluid and the **depth 深度** — the shape of the container does not matter.

Worked example. Find the extra pressure due to the water at the bottom of a swimming pool 2.5 m deep. (Water density 1000 kg m^{-3}, $g = 9.81$ m s^{-2}.)

$$\Delta p = \rho g \Delta h = 1000 \times 9.81 \times 2.5 \approx 2.5 \times 10^4 \text{ Pa.}$$

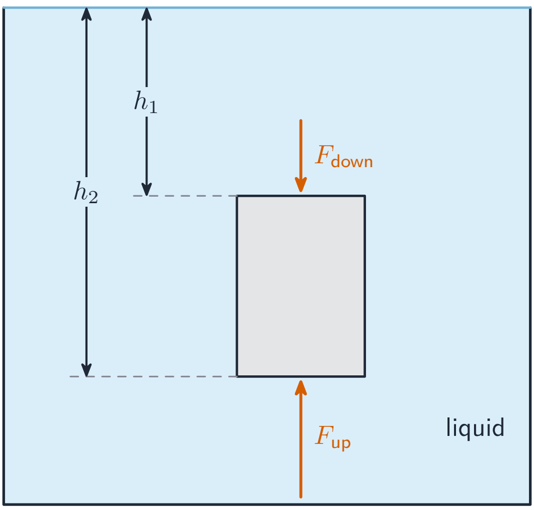


A column of liquid of area A: its weight sets the extra pressure at the depth below

For a submarine at depth h below the surface, the pressure from the water is $\rho_{\text{seawater}}\, g\, h$. For the total pressure, add the **atmospheric pressure 大气压强** at the surface (about 1.0×10^5 Pa).

Upthrust and Archimedes' principle

When an object is **submerged** 浸没 in a fluid, the pressure at the bottom of the object is greater than the pressure at the top (by $\rho g \Delta h$, where Δh is the object's height). This difference gives a net upward force called the **upthrust** 浮力.



Upthrust arises because the pressure on the bottom of the object is greater than on the top

For an object of volume V (the volume of fluid **displaced** 排开), the upthrust is

$$F_{\text{upthrust}} = \rho_{\text{fluid}}\, g\, V.$$

This is **Archimedes' principle** 阿基米德原理: the upthrust on a body in a fluid equals the weight of the fluid it pushes aside.

Worked example. A metal block of volume 2.0×10^{-3} m^3 is fully submerged in water. Find the upthrust on it. (Water density 1000 kg m^{-3}, $g = 9.81$ m s^{-2}.)

$$F_{\text{upthrust}} = \rho_{\text{fluid}}\, g\, V = 1000 \times 9.81 \times 2.0 \times 10^{-3} \approx 20 \text{ N}.$$

For a fully submerged object, V is its full volume. For a **floating** 漂浮 object, V is only the volume below the surface — the object floats when the upthrust on the part below the surface equals its weight.

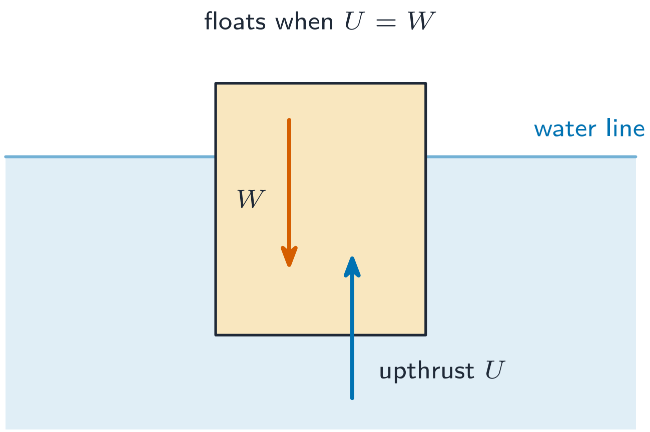
Whether an object floats at all depends on densities: for a fully submerged object the ratio of upthrust to weight is $\rho_{\text{fluid}} V g / \rho_{\text{object}} V g = \rho_{\text{fluid}} / \rho_{\text{object}}$, so it floats if it is less dense than the fluid and sinks if it is denser. Halving the object's density doubles that ratio; changing its volume or the depth changes nothing.

Worked example. A cylinder of mass 11 kg and diameter 0.78 m floats upright in water of density 990 kg m^{-3}. Find the depth y of its base below the surface.

Floating means upthrust = weight. The submerged volume is the base area times the depth, Ay, with $A = \pi \times 0.39^2 = 0.478$ m^2:

$$\rho g A y = mg \quad \Rightarrow \quad y = \frac{m}{\rho A} = \frac{11}{990 \times 0.478} = 0.023 \text{ m}.$$

Each extra newton of load on the cylinder needs an extra $1/(\rho g A)$ of depth, so the depth rises in a straight line with the added weight, wherever on the cylinder it is placed.



A floating object sinks until the upthrust on the submerged part equals its weight

Force balance with upthrust

A block held under water by a string tied to the bottom of the container is in equilibrium under three vertical forces: weight (down), **tension** 张力 (down), upthrust (up). Set $F_{\text{upthrust}} = W + T$ to find the tension.

A submerged block hanging from a **newton meter** 弹簧测力计 reads less than its weight in air, because of the upthrust: reading $= W - F_{\text{upthrust}}$.

Worked example. A cylinder of weight 25.0 N hangs from a newton meter fully submerged in water, and the meter reads 10.0 N. Find the volume of the cylinder.

The upthrust is the missing 15.0 N, and upthrust $= \rho g V$, so $V = 15.0/(1000 \times 9.81) = 1.53 \times 10^{-3}$ m^3. The same method finds the density of the cylinder: $25.0/(9.81 \times 1.53 \times 10^{-3}) = 1670$ kg m^{-3}.

The upthrust depends on the fluid density and the displaced volume, not on the object's material or depth (for an **incompressible** 不可压缩 fluid). On a planet with smaller g, the upthrust is smaller in the same ratio as the weight, so a floating object still floats with the same fraction below the surface.

Definitions the examiner accepts

A definition question is marked against fixed wording. Learn these exactly, and give one answer only.

Term	**Definition**
centre of gravity	the point at which the whole weight of an object may be considered to act
moment of a force	the product of the force and the perpendicular distance from the point to the line of action of the force
couple	a pair of equal and opposite forces whose lines of action do not coincide, which produces rotation only
torque of a couple	the product of one of the forces and the perpendicular distance between the lines of action of the forces
principle of moments	for a body in equilibrium, the sum of the clockwise moments about any point equals the sum of the anticlockwise moments about that point
equilibrium	the state of a body on which the resultant force and the resultant torque are both zero
density	mass per unit volume
pressure	force per unit area, where the force acts at right angles to the surface
upthrust	the upward force on a body in a fluid caused by the pressure being greater on its lower surface than on its upper surface
Archimedes' principle	the upthrust on a body in a fluid is equal to the weight of the fluid displaced by the body

Exam tips

- Take **moments** about a chosen pivot; for equilibrium, total clockwise = total anticlockwise moments **and** the resultant force is zero.
- A body in equilibrium under three forces gives a **closed triangle** of forces.
- Fluid pressure = $\rho g h$; **upthrust** = **weight of fluid displaced** (Archimedes).
- Distinguish mass, weight and density, and give the base unit each time.

Common mistakes

- Resolving with $\cos\theta$ where the geometry needs $\sin\theta$. Draw the triangle and check which side the angle is next to before choosing.
- Substituting the weight of the object where the question is about the upthrust on it. Upthrust is the weight of the fluid displaced, $\rho_{\text{fluid}}Vg$.
- Leaving out the weight of a uniform beam. It acts at the centre, and it has a moment about any pivot that is not at the centre.
- Using the distance along the rod for an angled force. The moment needs the perpendicular distance, or the perpendicular component of the force.
- Taking the torque of a couple as Fd with d measured from the pivot. The distance is between the two lines of action, so a symmetric couple gives $2Fd$.